matching theory

Matching can involve:

1. One-sided matching, i.e., the allocation or exchange of indivisible objects, such as dormitory rooms, transplant organs, courses, summer houses, etc.

2. Or matching can involve two-sided matching, in markets with two sides, such as firms and workers, students and schools, civil servants and positions, or men and women, that need to be matched with each other. Auctions can be seen as special cases of matching models, in which there is a single seller.

two-sided matching

1st reference

David Gale e Lloyd Shapley (1962):

“College admissions and the stability of marriage,”
American Mathematical Monthly, 69: 9-15

matching market

(M, W, P) where:

- M and W are two disjoint sets of agents
- P is a profile of strict orders of preferences (there are no utilities)

matching market

Notation:

- m is an element of M; w is an element of W; and v is an element of $M \cup W$
- $P_v \in \mathcal{P}_v$ is v's true order of preferences
- $P = (P_v)_{v \in M \cup W} \in \mathcal{P}$
- $\mathcal{M}$ is the set of matchings

example

M = {A,B,C}

W = {1,2,3}

A	B	C		1	2	3
1	2	2		A	B	C
2	1	1		B	C	A
3	3	3		C	A	B

matching

A matching μ in (M, W, P) is a one-to-one correspondence from M∪W onto itself of order 2 such that

- if $\mu(w) \neq w$, then $\mu(w) \in M$
- if $\mu(m) \neq m$, then $\mu(m) \in W$

example

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B

example

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	**B**	**C**	**A**
3	3	3	C	A	B

example

A	B	C		1	2	3
1	2	2		A	B	C
2	1	1		B	C	A
3	3	3		C	A	B

example

A	B	C		1	2	3
1	2	2		**A**	**B**	**C**
2	1	1		B	C	A
3	3	3		C	A	B

stability

(m,w) **block** μ iff m P_w μ(w) and w P_m μ(m).

A matching μ in (M, W, P) is **stable** iff

- it is individually rational (IR), i.e., μ(v) = v or μ(v) P_v v for all v ∈ M∪W and
- no pair of agents (m,w) blocks μ

stability

A stable matching always exists in a two-sided matching market.

Gale and Shapley algorithm

Step 1: each man proposes to the first woman on his list; each woman retains his best proposal and rejects all others (if they exist)

⋮

Step t: each rejected man in step t-1, proposes to the next woman on his list; each woman retains the best proposal (among this step's proposals and the previously retained man) and rejects the others (if they exist)

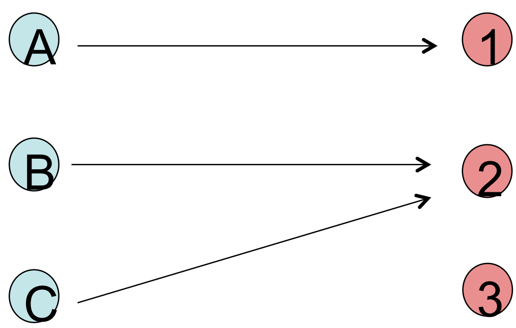
End: No man is rejected. Women are assigned to the man they retain.

example

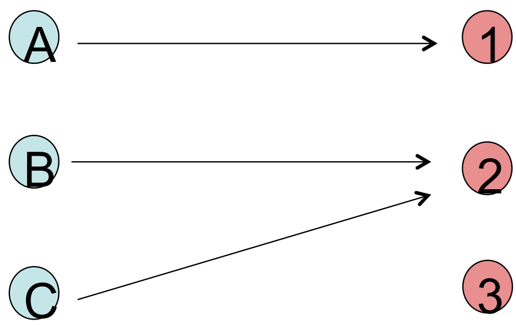
A	B	C		1	2	3
1	2	2		A	B	C
2	1	1		B	C	A
3	3	3		C	A	B

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B

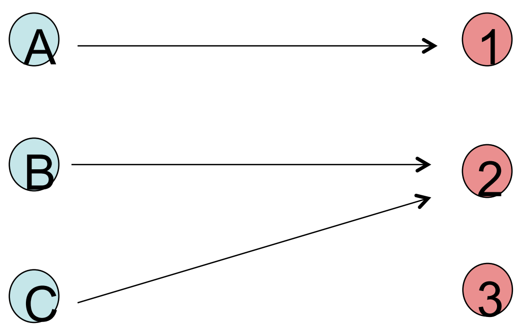


A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B



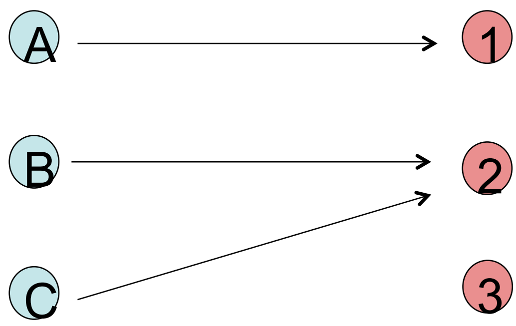
1	A
2	B C
3	

A	B	C	1	2	3
1	2	2	A	**B**	C
2	1	1	B	**C**	A
3	3	3	C	A	B



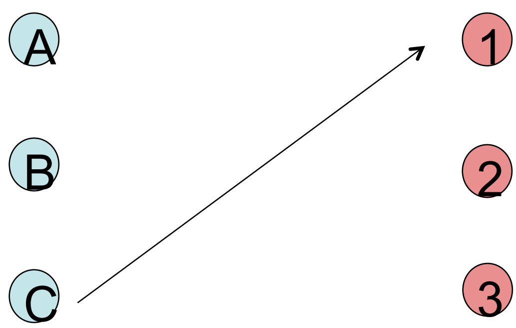
1	A
2	B C
3	

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B



1 | A
2 | B ~~C~~
3 |

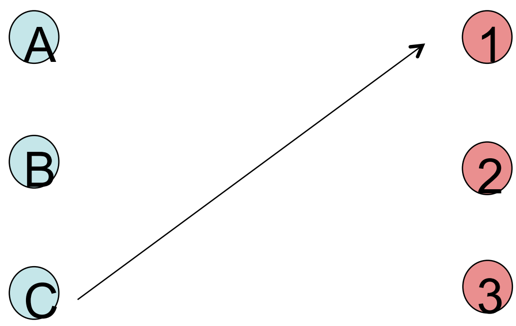
A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B



1 | A C
2 | B ~~C~~
3 |

A	B	C
1	2	2
2	1	1
3	3	3

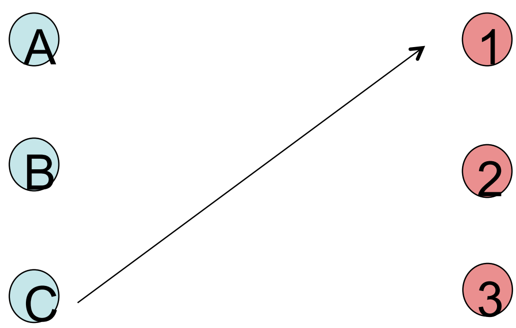
1	2	3
A	B	C
B	C	A
C	A	B



1 | A C
2 | B ~~C~~
3 |

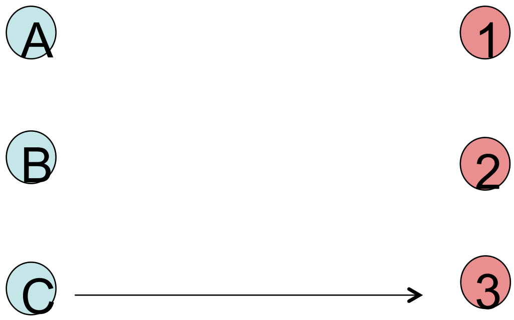
A	B	C
1	2	2
2	1	1
3	3	3

1	2	3
A	B	C
B	C	A
C	A	B



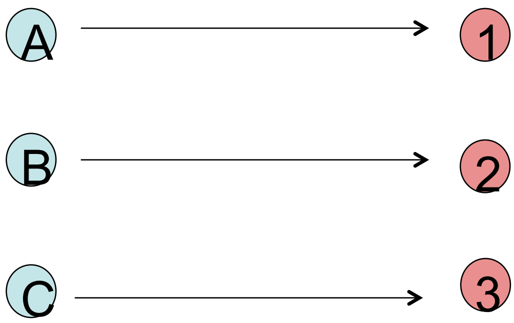
1 | A ~~C~~
2 | B ~~C~~
3 |

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B



1 | A ~~C~~
2 | B ~~C~~
3 | C

A	B	C	1	2	3
1	2	2	A	B	C
2	1	1	B	C	A
3	3	3	C	A	B



1 | A
2 | B
3 | C

Stable matching!

stability

In general, there may be several stable matchings.

The men-proposing G&S algorithm gives the stable matching which every man weakly prefers to any other stable matching (man-optimal stable matching).

Every man weakly prefers any stable matching to the woman-optimal stable matching.

The set of stable matchings is a lattice.

example

A	B	C	1	2	3
2	3	1	A	B	C
3	1	2	B	C	A
1	2	3	C	A	B

Three stable matchings:

1	C	1	B	1	A
2	A	2	C	2	B
3	B	3	A	3	C

matching mechanism

Given a matching market (M,W,P), a **direct matching mechanism** is a procedure that assigns a matching μ to each P, i.e. $\varphi: \mathcal{P} \rightarrow \mathcal{M}$.

A matching mechanism is

- IR iff it always produces IR matchings
- Stable iff it always produces stable matchings
- Pareto efficient (PE) iff it always produces PE matchings

matching mechanism

A **matching mechanism** can be centralized or decentralized

We will look at **centralized** matching mechanisms:

- all agents submit simultaneously lists of preferences to a matchmaker
- The matchmaker processes the submitted lists by means of an algorithm to produce a matching

game

A centralized matching mechanism induces a **simultaneous-move game G =** $(I, (S_i)_{i \in I}, (u_i)_{i \in I})$ where

- $I = M \cup W$
- $S_i = \mathcal{P}_i$, $i \in M \cup W$
- u_i: $\mathcal{M} \rightarrow \mathcal{R}$ is i's utility function over matchings; since one only cares about one's own match, it is derived from i's utility over potential partners – any utility function consistent with i's true list (order) of preferences $\mathcal{P}_i$

strategy-proofness

Obviously, given a matching mechanism, agents don't have to reveal their true preferences.

They may play strategically.

A matching mechanism is **strategy-proof** iff submitting the true lists of preferences is a dominant strategy.

stability and strategy-proofness

Strategy-proofness, IR, stability and PE are desirable properties.

But:

Theorem: There is exists no matching mechanism that is both stable and strategy-proof.

And we can always compute NE…

incentives

The mechanism that uses G&S algorithm is strategy-proof for the proposing side of the market.

But, if the market has more than one stable matching, there is at least one agent in the receiving side that can successfully manipulate.

example

A	B
1	2
2	1

1	2
B	A
A	B

In this matching market there are 2 stable matchings.

When G&S algorithm with letters proposing is applied, submitting the true preferences is a dominant strategy for letters.

But numbers have profitable deviations: for example, truncating the true preferences

Nash equilibria

Theorem: In the game induced by the G&S algorithm, μ is a Nash equilibrium (NE) outcome iff μ is IR.

Theorem: In the game induced by the G&S algorithm, μ is the outcome of a NE where one side uses its dominant strategy iff μ is stable.

extensions

Indifferences

Many-to-one matching

Many-to-many

Money

one-sided matching (resource allocation)

house allocation problem

A house allocation problem is a triple (A, H, P) where A is a set of n agents, H is a set of n houses, and P is a strict preference profile of agents over houses.

A matching is a one-to-one and onto function:

$$\mu : A \to H$$

housing market

A housing market is a list (A, $(a,h_a)_{a \text{ in } A}$, P) where A is a set of n agents, h_a is the house occupied by a, and P is a strict preference profile of agents over houses.

Shapley and Scarf, 1974: The core of a housing market is non-empty.

Top Trading Cycles (TTC)

Step 1: Let each agent point to her top choice house and each house point to its owner. In this graph there is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house that she is pointing to.

Step k: Let each remaining agent point to her top choice among the remaining houses and each remaining house point to its owner. There is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house that she is pointing to.

The algorithm terminates when no agents and houses remain. The assignments formed during the execution of the algorithm are the matching outcome.

housing market

The TTC gives the unique core allocation of a housing market.

The mechanism using TTC is IR, strategy-proof, and Pareto efficient.

housing market vs house allocation

Contrary to a what happens in housing markets, there is no perfect allocation mechanism in house allocation.

And mechanisms that are used in practice prioritize agents. Exs: random serial dictatorship (RSD)

hybrid model: house allocation with existing tenants

In this problem, there is a set of tenants occupying houses, but there are also newcomers. There is a set of vacant houses and there is a set of occupied houses. The problem is a five-tuple

$$(A_E, A_N, H_O, H_V, P)$$

This problem resembles the Kidney Exchange problem (considering both transplants from cadavers and from living donors)

Top trading cycles and chains

Consider a priority order of agents.

Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until someone requests the house of an existing tenant.

If at that point the existing tenant whose house is requested is already assigned another house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting the existing tenant before the requestor at the priority order and proceed with the first step of procedure through this existing tenant.

Top trading cycles and chains

Similarly, insert any existing tenant who is not already served just before the requestor in the priority order once her house is requested by an agent.

If at any point a cycle forms, it is formed by exclusively existing tenants and each of them requests the house of the tenant who is next in the cycle. (A cycle is an ordered list (h_{a1}, a_1,…h_{ak}, a_k) of occupied houses and existing tenants where agent a1 demands the house of agent a_2, h_{a2}, agent a_2 demands the house of agent a_3,… agent a_k demands the house of agent a1, h_{a1}.) In such cases, remove all agents in the cycle by assigning them the houses they demand and proceed similarly.

Top trading cycles and chains

The above algorithm is individually rational, Pareto efficient, and strategy-proof.