On Swap Rate Dynamics: To freeze or not to freeze?

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Abstract

We explore the implications of a common market and academic practice which is known as "freezing the drift" when dealing with swap interest rate dynamics.

In mathematical terms this can be better understood as imposing a low variance martingale (LVM) assumption. We look into the LVM Assumption implications, both on the shape and dynamics for default-free yield curves. We show that the LVM Assumption is equivalent to consider future yield curves are nothing but deterministic translations of the initial curve. For the particular case of the Nelson Siegel yield curve calibration, we show the LVM Assumption requires a deterministic parameter's evolution and, thus, imposes the need to constantly recalibrate the model.

Finally, based upon ECB historical data on evolution of the default-free Euro area yield curve, we illustrate periods in which the LVM may be applicable and others in which is not.

Keywords: drift freeze; low variance martingale; instantaneous forward rate; yield curve; Nelson Siegel model.

1 Introduction

It is hard to deal with the dynamics of swap interest rates, regardless of the chosen interest rate model. The difficulty arises because swap rate dynamics have an elaborate drift expression, under the martingale measures needed to compute prices or hedge most interest rate derivatives.

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The standard practice is, thus, "to freeze the drift" , i.e., to assume that some processes can be approximated by their initial values. Freezing the drift was first proposed by Brace et al. [6] for the pricing of swaptions under the context of the lognormal forward–Libor model or Libor market model (LMM) and has been used by several authors ever since. For some recent papers that use this assumption in various contexts, we refer to Beveridge and Joshi [1], Chen and Sandman [10], Gerzelak and Oosterlee [15] or Mahfoudhi [17].

Freezing the drift allows for great computational ease and to handle various fixed income products in a consistent way. Concretely, under this assumption, it is possible:

- to show that, for LMM, the swap interest rates are approximately lognormal, which result in the fusion of all interest rate market models (see Brace et al. [5] and d'Aspremont [11]);
- to extend the LMM to include more realistic forward volatilities (see Brace and Womersley [7]);
- to compute, for affine term structure models, closed-form swaption prices (see Schrager and Pelsser [22]) and convexity adjustments (see Gaspar and Murgoci [14]).

Despite the industry popularity and the growing literature which simply "freezes the drift", there are only few studies focusing on its financial implications, either from a theoretical point of view or from an empirical perspective.

The studies that do exist tend to look at the accuracy of the approximation that results from freezing the drift, for the pricing of a *particular product*, given the choice of a *particular interest rate model*. In the context of the LMM, some authors – see Rebonato [20], Brace et al. [4], Dun et al. [12], Schlögl [21] or Brigo and Liinev [8] – argue that freezing the drift can be justified. On the other hand, Kurbanmruradov et al. [16] shown that it does not yield acceptable results for exotic derivatives and long time horizons.

The existing results are limited because they focus on *particular* products and/or models. Although, "freezing the drift" may provide accurate price approximations for some products, it will not for others, such as exotics. Also, if we take a model – say the LMM as given – we are already assuming forward rates are lognormal and it could be that under that particular context, "freezing the drift" is not that bad. But what about in general? Or in real life? Is "freezing the drift" of swap rates a realistic assumption?

This paper contributes to the literature by answering the above questions and by taking a different perspective. We do not take into to account any particular interest rate product and/or model, instead we look into the assumption *per se*. We formally state it and analyse its implications in terms of the shape and dynamics of the default-free yield curve.

The remain of this paper is organised as follows. Section 2 briefly introduces the notation and our setup. It starts by reviewing some basic concepts on swap interest rates and the swap martingale measure. It then shows "freezing the drift" is equivalent to assuming some processes are low variance martingales (LVM) under the swap martingale measure and formally states the LVM Assumption. Section 3 presents the key theoretical results on the implications of the LVM to the shape and dynamics of default-free yield curves. In Section 4 we focus on the Nelson Siegel parameterization for calibrating yield curves and prove that the LVM implies constant (but deterministic) recalibration of the model. In Section 5 we take an empirical perspective. Based upon historical calibrations of the Euro zone default-free yield curve, preformed by the European Central Bank (ECB) we illustrate the (un)realism of having a LVM Assumption. We show that if it is true that for some particular instants the true yield curve and the yield curve implied by the LVM are relatively close to one another, most of the times they are not. Section 6 concludes the paper summarising the results and discussing their implications both for academic research and industry.

Let us consider a fixed-for-floating forward-start interest rate swap (IRS) with contract date t_0. The starting date of the first effective period of the swap is T_0 and the payment dates are T_1, T_2,..., T_N, with $t_0 < T_0 < T_1 < ... < T_N$. We define the tenor structure as $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ and the year's fractions between the tenor's dates as $\alpha_i = T_i - T_{i-1}$, for $i = 1, 2, ..., N$ (see Figure 1). Without loss of generality, we consider the nominal value equal to 1. For $i = 1, 2, ..., N$, the floating leg pays at each time T_i the amount $\alpha_i L(T_{i-1}, T_i)$, where the LIBOR rate $L(T_{i-1}, T_i)$ is observed at time T_{i-1} for the maturity T_i. The fixed leg pays the amount $\alpha_i S(t_0, \mathcal{T})$, where $S(t_0, \mathcal{T})$ is a fixed interest rate contracted at time t_0 for the tenor structure $\mathcal{T}$.

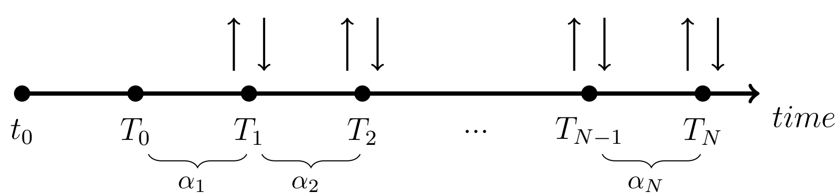

Figure 1: Forward-start interest rate swap's timeline.

Given a tenor $\mathcal{T}$ and for any date t before the first effective period T_0, i.e. $t_0 < t < T_0$ and $T_i \in \mathcal{T}$, the forward swap rate $S(t, \mathcal{T})$ is the fixed rate that makes the IRS a fair contract at that time t. We, thus, obtain

$$S(t, \mathcal{T}) = \frac{p(t, T_0) - p(t, T_N)}{P(t, \mathcal{T})}, \tag{1}$$

where $p(t, T)$ is the price at time t, of a zero coupon bond that pays 1 at time T_i for $T_i \in \mathcal{T}$, and $P(t, \mathcal{T})$ represents the price process of a portfolio of zero coupon bonds,

$$P(t, \mathcal{T}) = \sum_{i=1}^{N} \alpha_i p(t, T_i). \tag{2}$$

The portfolio price process, $\boldsymbol{P}(\mathcal{T}) = \{P(t, \mathcal{T}) : t_0 \leq t < T_0\}$, is known as the *swap level*.

Let us now consider the martingale measure where the *numeraire* is the swap level. This measure is usually called *swap measure* and we will represent it by $\mathbb{Q}^{S(\mathcal{T})}$, to reinforce the fact that it depends on the chosen tenor $\mathcal{T}$.

It becomes clear that the process of the swap rates $\boldsymbol{S}(\mathcal{T}) = \{S(t,\mathcal{T}) : t_0 \leq t < T_0\}$ is a martingale under the swap measure, i.e., for $t_0 \leq s < t < T_0$,

$$E^{\mathbb{Q}^{S(\mathcal{T})}}[S(t,\mathcal{T})|\mathcal{F}_s] = E^{\mathbb{Q}^{S(\mathcal{T})}}\left[\left.\frac{p(t,T_0)-p(t,T_N)}{P(t,\mathcal{T})}\right|\mathcal{F}_s\right] = \frac{p(s,T_0)-p(s,T_N)}{P(s,\mathcal{T})} = S(s,\mathcal{T}).$$

For further details on swap martingale measures we refer to Björk [2] or Brigo and Mercurio [9].

Let us now consider the ratio between each zero coupon bond price and the swap level, that is, for $i = 1,2,...,N$ and $t_0 \leq t < T_0$,

$$X_i(t,\mathcal{T}) = \frac{p(t,T_i)}{P(t,\mathcal{T})}. \tag{3}$$

We can then conclude that, the processes $\boldsymbol{X}_i(\mathcal{T}) = \{X_i(t,\mathcal{T}) : t_0 \leq t < T_0\}$ are also $\mathbb{Q}^{S(\mathcal{T})}$ - martingales. That is, $E^{\mathbb{Q}^{S(\mathcal{T})}}[X_i(t,\mathcal{T})|\mathcal{F}_s] = X_i(s,\mathcal{T})$, for each $i = 1,2,...,N$ and $t_0 \leq s < t < T_0$.

Furthermore, for $\alpha_i = \alpha$, for $i = 1,...,N$ – which is common in financial markets where 6 month or 1 year is the standard – and for $t_0 \leq t < T_0$, X_i can be further simplified to

$$X_i(t,\mathcal{T}) = \frac{1}{\alpha}\left(\frac{p(t,T_i)}{\sum_{j=1}^{N} p(t,T_j)}\right).$$

The ratio in brackets represents the proportion of the price of a zero coupon bond maturing at the payment date T_i, $p(t,T_i)$ over the sum of prices of all zero coupon bonds maturing at all payment dates $T_1,....,T_N$ of the swap under consideration.

For the purpose understanding why it is so important to freeze the drift when dealing with swap rates and its connection to the processes $\boldsymbol{X}_i(\mathcal{T}) = \{X_i(t,\mathcal{T}) : t_0 \leq t < T_0\}$, we have a quick look at the dynamics of the swap rates. We follow the approach in Gaspar and Murgoci [14] and take as given the risk-neutral dynamics of zero-coupon bond prices

$$dp(t,T) = r_t p(t,T)dt + v(t,T)p(t,T)dW_t^{\mathbb{Q}} \tag{4}$$

where r is the short interest rate, v is any adapted process and W is a Wiener process under the risk-neutral martingale measure denoted by $\mathbb{Q}$.

The dynamics in (4) is not restrictive as any stochastic interest rate model – no matter if it is a market model, a short rate model or and HJM-type model – imply some dynamics for zero coupon bonds and the key part of that dynamics is the volatility process v.

Given the dynamics in (4) and a concrete IRS with tenor $\mathcal{T}$, it is straightforward to derive the $\mathbb{Q}$-dynamics for the portfolio of bond prices that is swap level. Indeed, the dynamics of $P(t,\mathcal{T})$ for $t_0 \le t < T_0$ is given by,

$$dP(t,\mathcal{T}) = r_t P(t,\mathcal{T})\,dt + V(t,\mathcal{T})\,P(t,\mathcal{T})\,dW_t^{\mathbb{Q}} \ ,$$

where we have the notation

$$V(t,\mathcal{T}) = \sum_{i=1}^{N} \frac{\alpha_i p(t,T_i) v(t,T_i)}{P(t,\mathcal{T})} = \sum_{i=1}^{N} \alpha_i X_i(t,\mathcal{T}) v(t,T_i) \ ,$$

and the dependence on $X_i(t,\mathcal{T})$, defined in (3), becomes clear.

Using the above dynamics for zero coupon bond price and for swap level it is possible to derive, the swap rate $S(t,\mathcal{T})$. Under the swap measure, $\mathbb{Q}^{\mathcal{S}(\mathcal{T})}$, the swap rate dynamics is given by

$$dS(t,\mathcal{T}) = \{v(t,T_0)X_0(t,\mathcal{T}) - v(t,T_N)X_N(t,\mathcal{T}) - [X_0(t,\mathcal{T}) - X_N(t,\mathcal{T})]V(t,\mathcal{T})\}dW_t^{\mathcal{S}} \ .$$

Notice the diffusion term above is quite complex as $X_0(t,\mathcal{T})$ and $X_N(t,\mathcal{T})$ are given by (3).

Moreover, under any martingale measure, other than the swap measure, the drift of the swap rate dynamics also depends on the processes $\boldsymbol{X}_i(\mathcal{T}) = \{X_i(t,\mathcal{T}) : t_0 \le t < T_0\}$, making simulations and pricing of interest rate products intractable.

This is the reason many practitioners and researchers opt for "freezing the drift", which relies on considering

$$X_i(t,\mathcal{T}) \approx X_i(t_0,\mathcal{T}).$$

We start Section 2 by stating formally the assumption on the processes $\boldsymbol{X}_i(\mathcal{T})$ for $i = 1, 2, ..., N$.

2 Low Variance Martingale Assumption

Formally by "freezing the drift" one assumes the ratio between the price of a zero coupon bond and a portfolio of discount bonds is a *low variance martingale* (LVM) under the swap measure. Therefore, the ratio's value at each time may be approximated by its conditional expected value, particularly by its time zero value. The LVM Assumption can be enunciated as follows.

Assumption 2.1. *(LVM)* Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contractual date t_0, and let $\boldsymbol{X}_i(\mathcal{T}) = \{X_i(t,\mathcal{T}) : t_0 \le t < T_0\}$, for $i = 1, 2, ..., N$, be the processes, where $X_i(t,\mathcal{T})$ is defined in (3).

The processes $\boldsymbol{X}_i(\mathcal{T})$, for $i = 1, 2, ..., N$, are low variance $Q^{S(\mathcal{T})}$ - martingales, i.e., for $i = 1, 2, ..., N$ and $t_0 \le t < T_0$,

$$X_i(t,\mathcal{T}) \approx E^{Q^{S(\mathcal{T})}}(X_i(t,\mathcal{T})|\mathcal{F}_{t_0}) = X_i(t_0,\mathcal{T}).$$

Aiming at an interpretation of the LVM Assumption, we consider the case where $\alpha_i = \alpha, i = 1, ..., N$. The assumption states that $X_i(t, \mathcal{T}) \approx X_i(t_0, \mathcal{T}), i = 1, ..., N$, which means that for each time t, such that, $t_0 \leq t < T_0$, the proportion that the zero coupon bond with maturity T_i represents on the portfolio at time t is equal to the proportion at time t_0. In other words, we freeze the proportion at time t_0. To freeze is equivalent to consider that at each time t the zero coupon bond is obtained by multiplying the portfolio at this time by the initial proportion. That is, the zero coupon bond with maturity T_i for any time t corresponds always to the same proportion of the portfolio at this time.

In the rest of this section we study the implications of the LVM Assumption.

Let us introduce new simplifying notations,

$$D(t_0, t, T) = \frac{p(t, T)}{p(t_0, T)} \qquad \text{and} \qquad K(t_0, t, \mathcal{T}) = \frac{P(t, \mathcal{T})}{P(t_0, \mathcal{T})}, \tag{5}$$

with $t_0 \leq t < T_0$ and $T \in \mathcal{T}$, where $D(t_0, t, T)$ (respectively, $K(t_0, t, \mathcal{T})$) is the zero coupon bond price (respectively, portfolio price) at time t relative to its initial value.

Proposition 2.1 states our first LVM Assumption implication.

Proposition 2.1. *Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contract date t_0. The LVM Assumption is equivalent to*

$$D(t_0, t, T_i) \approx K(t_0, t, \mathcal{T}), i = 1, 2, ..., N$$

for $t_0 \leq t < T_0$, where $D(t_0, t, T_i)$ and $K(t_0, t, \mathcal{T})$ are defined in (5).

Proof. Suppose that the LVM Assumption holds, this means that for $t_0 \leq t < T_0$,

$$X_i(t, \mathcal{T}) \approx X_i(t_0, \mathcal{T}), i = 1, ..., N.$$

Recalling the $X_i(t, \mathcal{T})$ definition in (3), we can rewrite the LVM Assumption as

$$\frac{p(t, T_i)}{P(t, \mathcal{T})} \approx \frac{p(t_0, T_i)}{P(t_0, \mathcal{T})}, i = 1, 2, ..., N,$$

which is equivalent to

$$\frac{p(t, T_i)}{p(t_0, T_i)} \approx \frac{P(t, \mathcal{T})}{P(t_0, \mathcal{T})}, i = 1, 2, ..., N.$$

Finally, taking into account (5) we conclude the proof. □

Note that since $K(t_0, t, \mathcal{T})$ does not depend on T_i, then the Proposition 2.1 also means that $D(t_0, t, T_1) \approx D(t_0, t, T_2) \approx ... \approx D(t_0, t, T_N)$. In other words, the Proposition 2.1 shows that, for each tenor $\mathcal{T}$ and contract date t_0, the LVM Assumption is equivalent to assuming that $D(t_0, t, T_i)$ does not depend on T_i, for $i = 1, 2, ..., N$. That is, it implicitly states that all zero coupon bonds included in the portfolio have the same price at time t relative to its initial value, no matter their maturity. Furthermore, this value is equal to the portfolio price at time t relative to its initial value.

At this point we want to explore the influence of the LVM Assumption on instantaneous forward interest rates. Let us remember the relation between zero coupon bonds and the instantaneous forward interest rate

$$p(t,T) = \exp\left(-\int_t^T f(t,u)\ du\right), \tag{6}$$

where $f(t,u)$ represents the instantaneous forward interest rate at time t with maturity u, with $t \leq u \leq T$. Using the previous relation, we can rewrite $D(t_0,t,T)$, for $t_0 \leq t < T_0$ and $i = 1,2,...,N$, as

$$D(t_0,t,T_i) = \exp\left(\int_{t_0}^t f(t_0,u)\ du\right) \exp\left(\int_t^{T_i} [f(t_0,u) - f(t,u)]\ du\right). \tag{7}$$

Corollary 2.1. *Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contract date t_0. If the LVM Assumption holds, instantaneous forward interest rate with different contract dates but with the same maturity have approximately the same value. Namely,*

$$f(t,T) \approx f(t_0,T), \quad t_0 \leq t < T_0, \quad T \in [T_1, T_N].$$

Proof. Considering $t \in [t_0, T_0[$ and $i,j \in \{1,2,...,N\}$ with $i < j$, if the LVM Assumption holds, by the Proposition 2.1, $D(t_0,t,T_i) \approx D(t_0,t,T_j)$. Taking into account (7), this is equivalent to

$$\int_t^{T_i} [f(t_0,u) - f(t,u)]\ du \approx \int_t^{T_j} [f(t_0,u) - f(t,u)]\ du,$$

which implies that

$$\int_{T_i}^{T_j} [f(t_0,u) - f(t,u)]\ du \approx 0, \quad \forall i,j \in \{1,2,...,N\}, i<j, \quad t_0 \leq t < T_0. \tag{8}$$

Considering $g(u) = f(t_0,u) - f(t,u)$, for $u \in [T_1, T_N]$, and omitting the dependence on t_0 and t, Equation (8) is equivalent to

$$\int_{T_i}^{T_j} g(u)\ du \approx 0, \quad \forall i,j \in \{1,2,...,N\}, i<j.$$

The function g is the difference of two continuous functions, then g is still continuous.

- If $g(u) \geq 0$ or $g(u) \leq 0$ for $u \in [T_i, T_j]$, given that $\int_{T_i}^{T_j} g(u)\ du \approx 0$, then $g(u) \approx 0$, for $u \in [T_i, T_j]$.

- If g does not have the same signal in the interval $[T_i, T_j]$, there are intervals where g is positive, negative or null.

  Without loss of generality, let us suppose that there is $S \in [T_i, T_j]$, such that $g(S) = 0$, with $g(u) \geq 0, u \in [T_i, S]$ and $g(u) \leq 0, u \in [S, T_j]$. We assume that there is a IRS, contracted at time t_0, with a tenor which only differ from the original IRS on one date. That is, the new IRS has one more payment date at time S. Then, using the same reasoning,

$$\int_{T_i}^{S} g(u)\, du \approx 0 \quad \text{and} \quad \int_{S}^{T_j} g(u)\, du \approx 0,$$

  which, along with $g(u) \geq 0, u \in [T_i, S]$ and $g(u) \leq 0, u \in [S, T_j]$, results in $g(u) \approx 0$ for $u \in [T_i, T_j]$.

We conclude that $g(u) \approx 0$, for $u \in [T_i, T_j], \quad \forall i, j \in \{1, 2, ..., N\}, i < j$, which implies that $g(u) \approx 0, u \in [T_1, T_N]$. Remembering the definition of g function, we obtain $f(t_0, T) - f(t, T) \approx 0$, for $T \in [T_1, T_N]$, which completes the proof. □

The previous corollary says that, under the LVM Assumption, the instantaneous forward rate depends only on the maturity and not on the date where it is evaluated. This implies that we obtain equal instantaneous forward rates for very different lengths of time to maturity. For instance, let us consider a 5 years annual forward-start IRS contract today that start in 1 year. The LVM Assumption implies that the instantaneous forward interest rate today for 2 years is the same in 3 months for 1 year and 9 months and the same in 6 months for 1 year and half, and so on. In general, it is not expected to observe this type of restrictive relations in the real market.

Using the definition of the forward swap rate and the relation between the zero coupon bond, the spot interest rate and the instantaneous forward interest rate, we can obtain the following result.

Corollary 2.2. *Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contractual date t_0. If the LVM Assumption holds, the forward swap rate at time t is approximately equal to the forward spot rate at time t_0, i.e. for $t_0 \leq t < T_0$*

$$S(t, \mathcal{T}) \approx S(t_0, \mathcal{T}).$$

Proof. Given Equation (6) and Corollary 2.1, we obtain for $t_0 \leq t \leq T$ and $T \in \mathcal{T}$, $p(t_0, T) \approx p(t_0, t) \times p(t, T)$. Using this equality and Equations (1) and (2) we conclude the proof. □

Based on this corollary we concluded that the forward swap rate at time t is determined by the forward swap rate at time t_0. Once again, this is not an expected behaviour on real life markets.

Now, we move into the implications of the LVM Assumption for the shape and dynamics of the term structure of instantaneous forward rates. For this, we need to compare several instantaneous forward rates, with different contract dates, but with the same time to maturity. In order to do that, it is convenient to use the Musiela [18] parameterization, in which $f_t(\tau)$ represents the instantaneous forward rate with time to maturity τ and contract date t.

Corollary 2.3. *Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contractual date t_0. If the LVM Assumption holds,*

$$f_t(\tau) \approx f_{t_0}(\tau + (t - t_0)), \quad t_0 \leq t < T_0, \quad T_1 - t < \tau < T_N - t.$$

In other words, the graphic of f_t is approximately obtained from the graphic of f_{t_0} by a horizontal translation associated to the vector $\vec{v} = (-(t - t_0), 0)$.

Proof. Using the Musiela's notation, we notice that the Corollary 2.1 can be written as $f_t(T-t) \approx f_{t_0}(T - t_0)$, for $t_0 \leq t < T_0$ and $T_1 \leq T \leq T_N$. Applying the change of variable $\tau = T - t$, we obtain $f_t(\tau) \approx f_{t_0}(\tau + (t - t_0))$, with $\tau \in]T_1 - t, T_N - t[$. $\square$

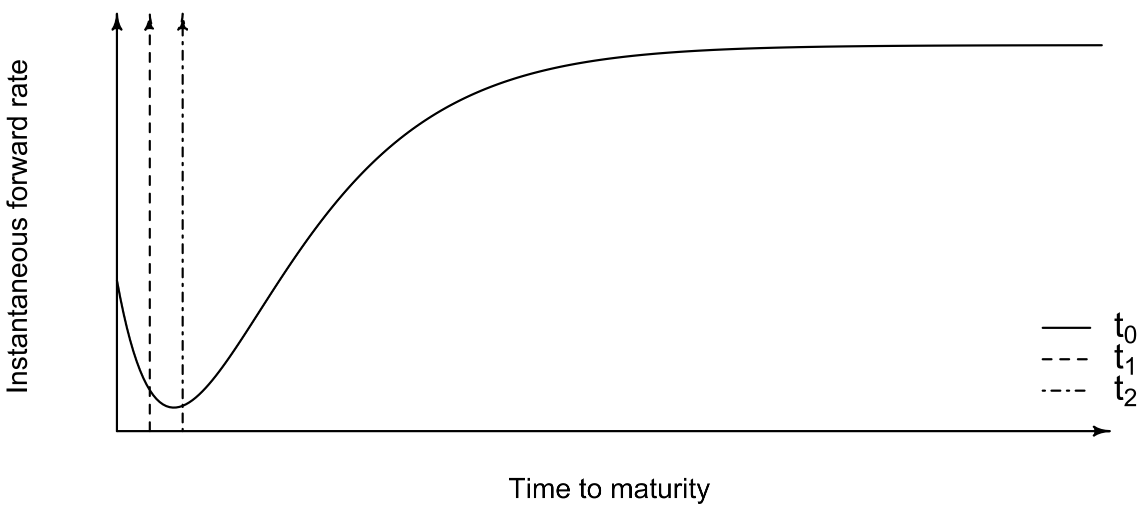

Figure 2: Yield curves at times t_0, t_1 and t_2, assuming that the LVM assumption holds for a forward-start IRS with contractual date t_0 and initial date T_0, with $t_0 < t_1 < t_2 < T_0$.

With this graphical approach it is clear that the LVM Assumption freezes the instantaneous interest rate curve at time t_0. The following curves, until the swap starts, are part of the first curve. In an informal way, we can look at it as the vertical axis moving to the right while the curve stays "frozen", as it is illustrated in Figure 2. Clearly, this is not an acceptable supposition because it means that at time t_0 we know a portion of the curve in future times.

3 Nelson Siegel Model

Nelson and Siegel [19] (NS) suggested a well known parameterization for the calibration of the instantaneous forward rates curve. Their parameterization takes instantaneous forward rates at any time t as linear combinations of three components.

Concretely, the NS parameterization is given by

$$f_t(\tau, b_t) = \beta_{0,t} + \beta_{1,t} \exp\left(-\frac{\tau}{\lambda_t}\right) + \beta_{2,t}\left(\frac{\tau}{\lambda_t}\right)\exp\left(-\frac{\tau}{\lambda_t}\right), \quad \tau > 0,$$

which can be rewritten as follows

$$f_t(\tau, b_t) = \beta_{0,t} + \left[\beta_{1,t} + \frac{\tau}{\lambda_t}\beta_{2,t}\right]\exp\left(-\frac{\tau}{\lambda_t}\right), \quad \tau > 0.$$

At any moment t calibration of this parameterization to market data requires optimising the vector of parameters $b_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t}, \lambda_t)$, with $\lambda_t > 0$. Therefore, we opt for notation $f_t(., b_t)$ instead of $f_t(.)$.

When the time to maturity approaches zero, the forward rate approaches the constant $\beta_{0,t}+\beta_{1,t}$,

$$\lim_{\tau \to 0^+} f_t(\tau, b_t) = \beta_{0,t} + \beta_{1,t}$$

and when the time to maturity approaches infinity, the forward rate approaches the constant $\beta_{0,t}$,

$$\lim_{\tau \to +\infty} f_t(\tau, b_t) = \beta_{0,t},$$

which means that the line $y = \beta_{0,t}$ is a horizontal asymptote of the graph of f_t, for large values of τ.

We determine the first and the second derivatives with respect to τ, respectively,

$$f_t'(\tau, b_t) = -\frac{1}{\lambda_t}\left[\beta_{1,t} + \beta_{2,t}\left(\frac{\tau}{\lambda_t} - 1\right)\right]\exp\left(-\frac{\tau}{\lambda_t}\right)$$

and

$$f_t''(\tau, b_t) = \frac{1}{{\lambda_t}^2}\left[\beta_{1,t} + \beta_{2,t}\left(\frac{\tau}{\lambda_t} - 2\right)\right]\exp\left(-\frac{\tau}{\lambda_t}\right).$$

If $\beta_{2,t} = 0$, once $\lambda_t > 0$, the shape of f_t's graph only depends on $\beta_{1,t}$. In particular,

- if $\beta_{1,t} = 0$, then f_t is a constant function equal to $\beta_{0,t}$, i.e, the term structure is flat;
- if $\beta_{1,t} > 0$, then f_t is a decreasing function with concave upwards;
- if $\beta_{1,t} < 0$, then f_t is an increasing function with concave downwards.

If $\beta_{2,t} \neq 0$, the monotony of f_t depends on the signal of $-\frac{\beta_{2,t}}{\lambda_t}\tau + (\beta_{2,t} - \beta_{1,t})$ and whenever $\frac{\beta_{1,t}}{\beta_{2,t}} < 1$ the extreme is attained at $\tau^* = \lambda_t\left(1 - \frac{\beta_{1,t}}{\beta_{2,t}}\right)$. Furthermore, the f_t's concavity depends on the signal of $\frac{\beta_{2,t}}{\lambda_t}\tau + (\beta_{1,t} - 2\beta_{2,t})$ and whenever $\frac{\beta_{1,t}}{\beta_{2,t}} < 2$ the inflection point is attained at $\tau^{**} = \lambda_t\left(2 - \frac{\beta_{1,t}}{\beta_{2,t}}\right)$.

We analyze the different graphs of f_t, depending on the relation between $\beta_{1,t}$ and $\beta_{2,t}$. For instance, for $\beta_{2,t} > 0$ or $\beta_{2,t} < 0$, respectively,

- if $\frac{\beta_{1,t}}{\beta_{2,t}} < 1$, the f_t's graph has a hump shape or a U shape;
- if $1 \leq \frac{\beta_{1,t}}{\beta_{2,t}} < 2$, the f_t is a decreasing or an increasing function;
- if $\frac{\beta_{1,t}}{\beta_{2,t}} \geq 2$, the f_t is a decreasing function with concave upwards or an increasing function with concave downwards.

The previous analysis highlights the different shapes the term structure can exhibit, depending on the chosen parameters. Each parameter can be associated with an economic interpretation: $\beta_{0,t}$ is a level parameter - the long term rate; $\beta_{1,t}$ is a slope parameter - the spread short/long term; $\beta_{2,t}$ is a curvature parameter; λ_t is a scale parameter.

Until now we have been studying the specificities of the NS model. At this moment we will investigate the LVM consequences in this model.

Proposition 3.1. *Let $\mathcal{T} = \{T_0, T_1, ..., T_N\}$ be a tenor associated to a forward-start IRS with contract date t_0. If the LVM Assumption holds, the relation between the vectors of parameters in NS model at times t_0 and t, $b_{t_0} = (\beta_{0,t_0}, \beta_{1,t_0}, \beta_{2,t_0}, \lambda_{t_0})$ and $b_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t}, \lambda_t)$, with $t_0 < t < T_0$, is given by*

$$\begin{cases} \lambda_t = \lambda_{t_0} \\ \beta_{0,t} = \beta_{0,t_0} \\ \beta_{1,t} = \left(\beta_{1,t_0} + \beta_{2,t_0}\frac{t-t_0}{\lambda_{t_0}}\right) e^{-\frac{t-t_0}{\lambda_{t_0}}} \\ \beta_{2,t} = \beta_{2,t_0} e^{-\frac{t-t_0}{\lambda_{t_0}}} \end{cases}$$

Proof. Considering t_0 and t, such that, $t_0 < t < T_0$, and $b_{t_0} = (\beta_{0,t_0}, \beta_{1,t_0}, \beta_{2,t_0}, \lambda_{t_0})$, from the NS model we have

$$f_{t_0}(\tau, b_{t_0}) = \beta_{0,t_0} + \left[\beta_{1,t_0} + \beta_{2,t_0}\frac{\tau}{\lambda_{t_0}}\right] \exp\left(-\frac{\tau}{\lambda_{t_0}}\right), \tau > 0. \tag{9}$$

On the one hand, due to the Corollary 2.3 we know that

$$f_t(\tau) = f_{t_0}(\tau + (t - t_0)), \forall \tau \in]T_1 - t, T_N - t[,$$

which implies that, for $\tau \in]T_1 - t, T_N - t[$,

$$f_t(\tau, b_t) = \beta_{0,t_0} + \left[\left(\beta_{1,t_0} + \beta_{2,t_0}\frac{t-t_0}{\lambda_{t_0}}\right) e^{-\frac{t-t_0}{\lambda_{t_0}}} + \left(\beta_{2,t_0} e^{-\frac{t-t_0}{\lambda_{t_0}}}\right)\frac{\tau}{\lambda_{t_0}}\right] \exp\left(-\frac{\tau}{\lambda_{t_0}}\right). \tag{10}$$

On the other hand, considering $b_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t}, \lambda_t)$ and calibrating the model at time t, we obtain

$$f_t(\tau, b_t) = \beta_{0,t} + \left[\beta_{1,t} + \beta_{2,t}\frac{\tau}{\lambda_t}\right] \exp\left(-\frac{\tau}{\lambda_t}\right), \tau > 0. \tag{11}$$

Note that the expressions referred on (10) and (11) have the same functional form, but with different parameters. Moreover, they coincide in the open set $]T_1 - t, T_N - t[$. This means that the two functions overlap in a open set. Using the result proven in the Appendix, we conclude that they must overlap in all domain and we obtain the system in (9). □

Proposition 3.1 states a very strong result. It shows that for *all* times t, with $t_0 < t < T_0$, the parameters of the NS model are nothing but deterministic functions of its parameters at time t_0. Moreover, these deterministic functions are non linear functionals of the original NS components. We get:

- the scale parameter is equal at t_0 and at t, $\lambda_t = \lambda_{t_0}$;
- the level parameter is equal at t_0 and at t, $\beta_{0,t} = \beta_{0,t_0}$;
- the slope parameter at t, $\beta_{1,t}$, has the same signal of $\beta_{1,t_0} + \beta_{2,t_0}\frac{t-t_0}{\lambda_{t_0}}$;
- the curvature parameter at t, $\beta_{2,t}$, has the same signal of the curvature parameter at t_0, β_{2,t_0}.

Taking into account the previous analysis about the shape of the curve f_t depending on the ratio $\frac{\beta_{1,t}}{\beta_{2,t}}$ and noting that $\frac{\beta_{1,t}}{\beta_{2,t}} = \frac{\beta_{1,t_0}}{\beta_{2,t_0}} + \frac{t-t_0}{\lambda_{t_0}}$, we conclude that f_t and f_{t_0} graphs can exhibit different shapes, depending on the value of $\frac{t-t_0}{\lambda_{t_0}}$. For larger values of t and lower values of λ_{t_0}, we perchance can observe different shapes.

The results in this section imply that "freezing the drift" requires constant recalibration of the yield curves, to keep having default-free yield curves within the NS family. Alternatively, using the consistency concept of Björk and Gaspar [3], we show the NS parameterization is *inconsistent* with the LVM Assumption.

4 Market Data

In this section, we empirically explore the LVM Assumption using real market data on the Euro zone default-free yield curve. The data is provided by the European Central Bank (ECB) on a daily basis, since mid 2004.

Our main goal is to identify some dates where the LVM Assumption may be reasonable and others in which the LVM Assumption is not acceptable.

We consider forward-start IRS with initial date, T_0, 1 year after the contract date t_0. We present yield curves on four different dates: 3 months, 6 months, 9 months and 1 year after the contract date. Whenever applied, we consider the first business day after the day we were looking for. We represent the yield curve until 15 years, to include swaps with possible long final dates. We use the same rate scale (form 0% to 6%) in all figures, for comparison purposes.

The following graphs compare for the dates referred in the graph title, the ECB constructed yield curve (full line) and the yield curve estimated by us using the LVM Assumption (dashed line).

We show the yield curves based on the annual forward-start IRS with initial dates at 01-Oct-2004, 02-May-2008 and 02-Jan-2013.

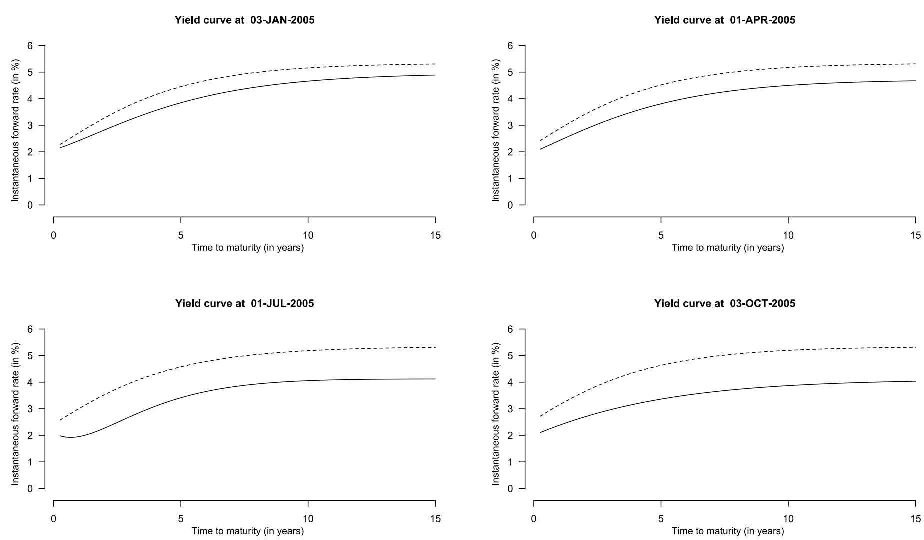

Figure 3: Yield curves at 03-Jan-2005, 01-Apr-2005, 01-Jul-2005 and 03-Oct-2005 based on the annual forward-start IRS with contract date at 01-Oct-2004. Full line: ECB yield curve at each date in the title; Dashed line: Yield curve at each date in the title, considering the LVM Assumption holds at 01-Oct-2004.

For the IRS with initial date at 01-Oct-2004 (see Figure 3), we observe that for the first dates the yield curves are closer than for the latest dates.

For the IRS with initial date at 02-May-2008 (see Figure 4), it seems that the LVM Assumption is not applicable. Although at 01-Aug-2008 the yield curves are not so different, all the remaining dates have very large differences. Actually, the shape of the yield curve from 2008 to 2009 changed substantially.

For the IRS with initial date at 02-Jan-2013 (see Figure 5), the LVM Assumption seems to be applied for all dates. Probably because the shape of the yield curve during 2013 has not changed

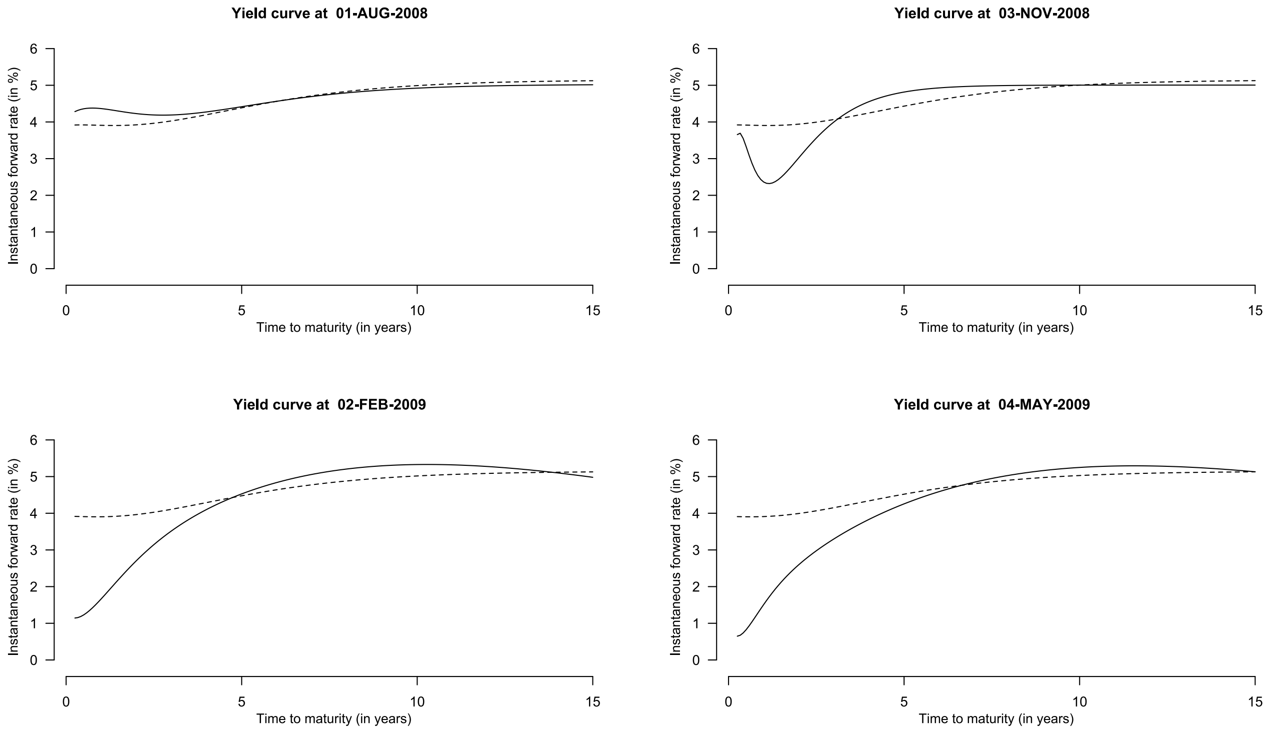

Figure 4: Yield curves at 01-Aug-2008, 03-Nov-2008, 02-Feb-2009 and 04-May-2009 based on the annual forward-start IRS with contract date at 02-May-2008. Full line: ECB yield curve at each date in the title; Dashed line: Yield curve at each date in the title, considering the LVM Assumption holds at 02-May-2008.

much.

At this point, we need a way to calculate the distance between the two term structures in each date, in order to compare extensively the differences along the time. Since we are comparing series of the same length, we decided to use the L_1 Minkowski distance.

Recall (for instance, consult Dunford and Schwartz [13]) that the L_p Minkowski distance between two time series $X_t = (x_1, x_2, ..., x_m)$ and $Y_t = (y_1, y_2, ..., y_m)$ is given by

$$d_{MIK}(X_t, Y_t) = \left(\sum_{i=1}^{m} |x_i - y_i|^p \right)^{\frac{1}{p}}.$$

When $p = 1$ it is the Manhattan distance and when $p = 2$ it is the Euclidean distance.

For each contract date represented with four graphs, we only consider the two first graphs, respectively 3 and 6 months after the contractual date. Further, do to the calculations we only consider a 5 years semiannual forward-start IRS. This means that $\alpha_i = T_0 - t_0 = 0.5$, for $i = 1, ..., N$ and $T_N - T_0 = 5$.

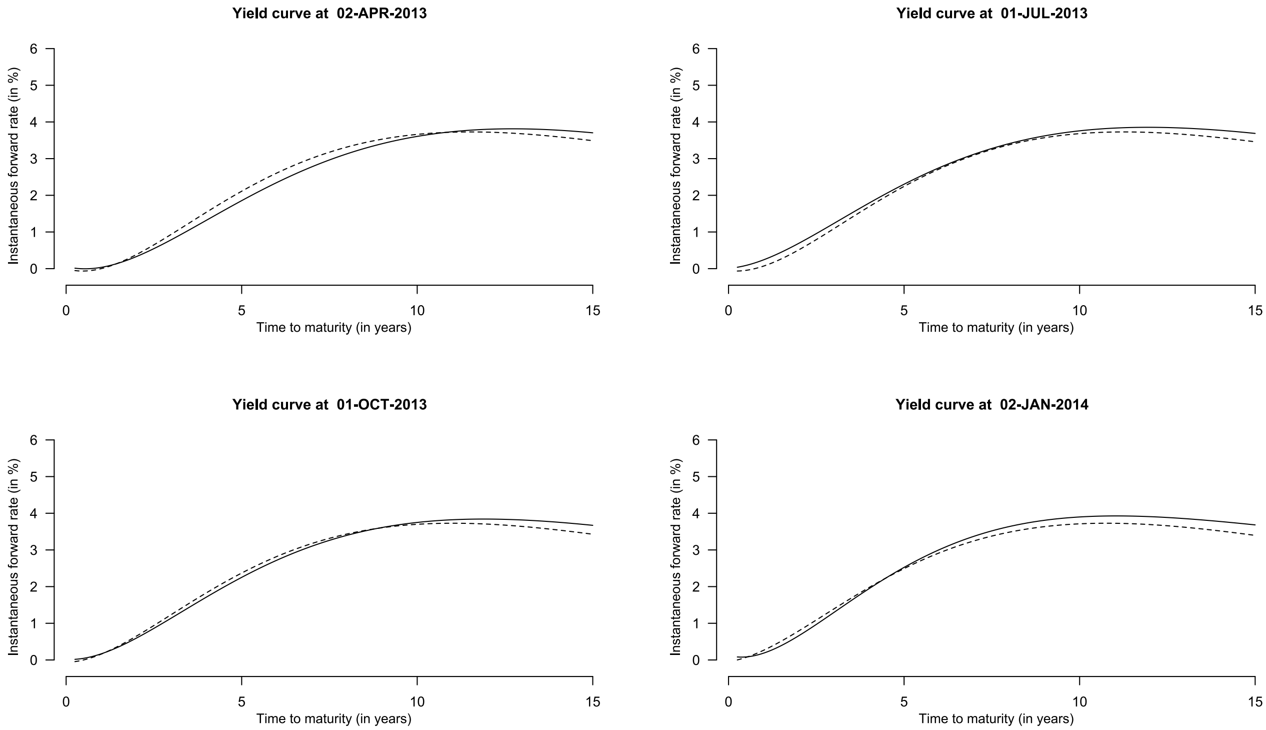

Figure 5: Yield curves at 02-Apr-2013, 01-Jul-2013, 01-Oct-2013 and 02-Jan-2013 based on the annual forward-start IRS with contract date at 02-Jan-2013. Full line: ECB yield curve at each date in the title; Dashed line: Yield curve at each date in the title, considering the LVM Assumption holds at 02-Jan-2013.

For each contract date we calculated the distance between the two time series:

- LVM series - the expected yield curve if the LVM Assumption holds for a 5 years semiannual forward-start IRS;
- ECB series - the yield curve obtained from the ECB for 3 (and 6) months after the contractual date.

Table 1 summarises the values obtained.

Contractual Date	3 months	6 months
01-Oct-2004	5.191	6.056
02-May-2008	5.167	12.834
02-Jan-2013	1.146	5.278

Table 1: L_1 Minkowski distance between the LVM serie and the ECB serie (for 3 and 6 months).

We also calculate the refereed distances for each contract date from 06 September 2004 until 15 October 2013. We obtained two graphs plotted in Figure 6, one for 3 months (full line) and other for 6 months (dashed line). Obviously, the values in each column of Table 1 are points in each graph.

The obtained graphs are represented in the Figure 6. The two graphs show that the distances are greater in times in which the yield curve have changed much. This is intrinsically related with the fact that the LVM Assumption freezes the yield curve.

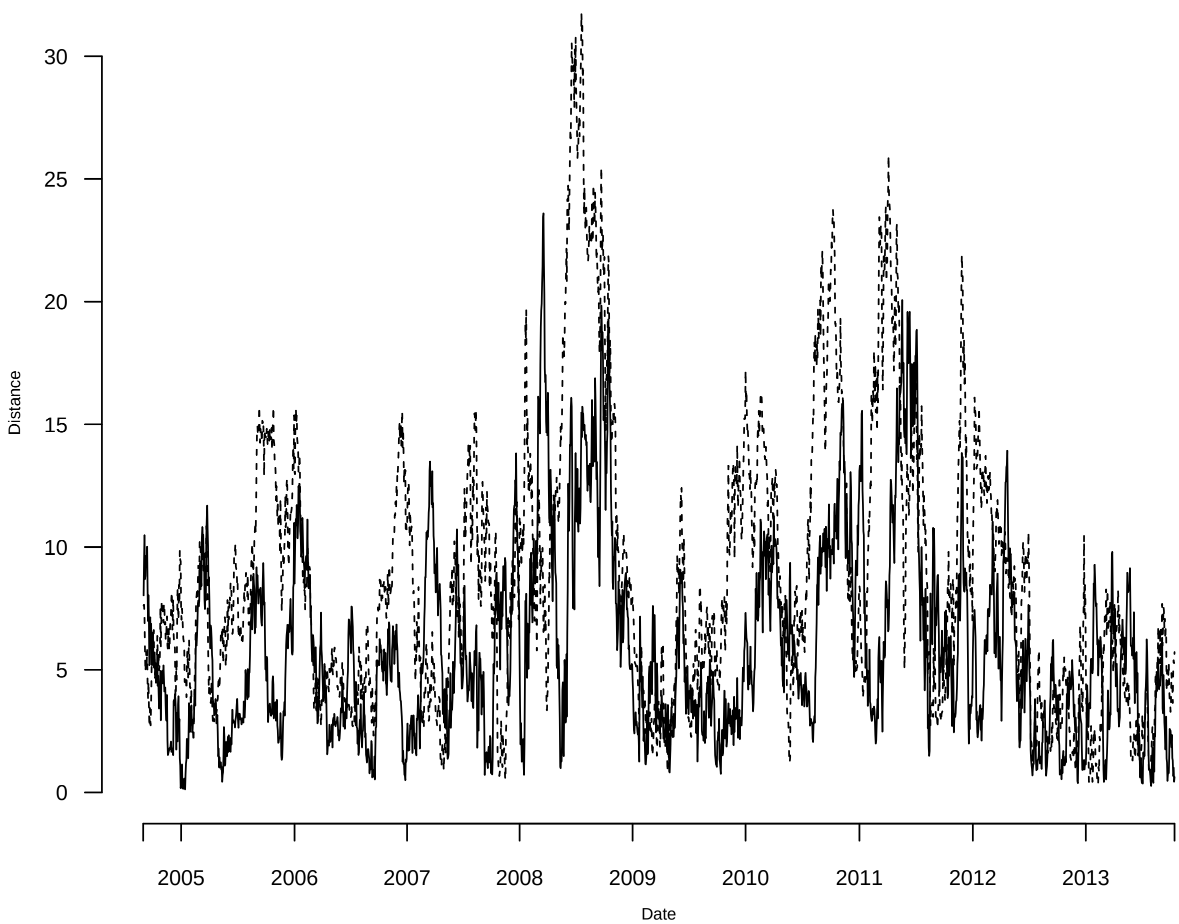

Figure 6: L_1 Minkowski distances between the two time series (LVM and ECB), for contractual dates from 06 September 2004 until 15 October 2013. Full line: 3 months; Dashed line: 6 months.

5 Conclusion

In this paper, we explore the common practice of "freezing the drift" when dealing with the dynamics of swap interest rates, for the purpose of simulations and pricing or hedging of interest rate derivatives that have swap rates as underlying.

We show that "freezing the drift" is equivalent to assuming that ratio between of zero coupon bond prices (of all maturities) and the swap level is a low variance martingale (LVM), under the swap martingale measure.

Contrary to the existing literature, we do not assume any particular stochastic interest rate model or financial product. We focus, instead, on the general implications of the LVM Assumption on the shape and dynamics of the default-free yield curve.

We have shown that the LVM Assumption means that the yield curve is frozen from the contract date of an IRS to the beginning of its first effective period. Consequently, we show the LVM Assumption imposes a deterministic but non linear evolution on the parameters of the popular NS parameterization, which implies inconsistency between the LMV assumption and this parameterization.

Finally, our empirical analysis of the LVM Assumption based upon historical Euro area default-free yield curves allowed us to identify since 2004, periods where LVM Assumption could be applicable and others in which this was definitely not the case.

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Appendix A.

Theorem. *Functions f and g have domain $\mathbb{R}$ and functional forms*

$$f(x) = a + (b + cx)\exp(dx), x \in \mathbb{R} \tag{12}$$

and

$$g(x) = a_1 + (b_1 + c_1 x)\exp(d_1 x), x \in \mathbb{R}. \tag{13}$$

If they overlap in an open set $I \subseteq \mathbb{R}$, then they overlap in $\mathbb{R}$, that is,

$$f(x) = g(x), x \in I \implies f(x) = g(x), x \in \mathbb{R}.$$

Proof. Let us consider functions f and g, with functional forms (12) and (13), and an open set $I \subseteq \mathbb{R}$. Let us suppose that $f(x) = g(x), x \in I$.

Calculating the first derivative of f we obtain

$$\begin{aligned} f'(x) &= c\exp(dx) + d(b + cx)\exp(dx) \\ &= c\exp(dx) + df(x) - ad. \end{aligned}$$

Note that

$$c\exp(dx) = f'(x) - df(x) + ad. \tag{14}$$

Using the previous derivative and the equality (14), we calculate the following derivatives

$$\begin{aligned} f''(x) &= cd\exp(dx) + df'(x) \\ &= f'(x) - df(x) + ad + df'(x) \end{aligned}$$

and

$$\begin{aligned} f'''(x) &= f''(x) - df'(x) + df''(x) \\ &= f''(x) + d(f''(x) - f'(x)). \end{aligned}$$

Since $f(x) = g(x)$ for $x \in I$, their derivatives also coincide in this interval,

$$f(x) = g(x) \wedge f'(x) = g'(x) \wedge f''(x) = g''(x) \wedge f'''(x) = g'''(x), x \in I$$

Simplifying separately each equation, we obtain

$$\begin{aligned} f'''(x) = g'''(x), x \in I &\Leftrightarrow f''(x) + d(f''(x) - f'(x)) = g''(x) + d_1(g''(x) - g'(x)) \\ &\Leftrightarrow f''(x) + d(f''(x) - f'(x)) = f''(x) + d_1(f''(x) - f'(x)) \\ &\Leftrightarrow (f''(x) - f'(x))(d - d_1) = 0 \\ &\Leftrightarrow d_1 = d \end{aligned}$$

Using the fact that $d_1 = d$, we have

$$\begin{aligned} f''(x) = g''(x), x \in I \quad &\Leftrightarrow \quad f'(x) - df(x) + ad + df'(x) = g'(x) - d_1 g(x) + a_1 d_1 + d_1 g'(x) \\ &\Leftrightarrow \quad f'(x) - df(x) + ad + df'(x) = f'(x) - df(x) + a_1 d + df'(x) \\ &\Leftrightarrow \quad a_1 = a \end{aligned}$$

Using the equalities $a_1 = a$ and $d_1 = d$, we conclude that

$$\begin{aligned} f'(x) = g'(x), x \in I \quad &\Leftrightarrow \quad c\exp(dx) + df(x) - ad = c_1 \exp(d_1 x) + d_1 g(x) - a_1 d_1 \\ &\Leftrightarrow \quad c\exp(dx) + df(x) - ad = c_1 \exp(dx) + df(x) - ad \\ &\Leftrightarrow \quad c_1 = c \end{aligned}$$

Finally, using $a_1 = a$, $c_1 = c$ and $d_1 = d$, we prove that

$$\begin{aligned} f(x) = g(x), x \in I \quad &\Leftrightarrow \quad a + (b + cx)\exp(dx) = a_1 + (b_1 + c_1 x)\exp(d_1 x) \\ &\Leftrightarrow \quad b_1 = b \end{aligned}$$

Summarizing $a = a_1 \wedge b = b_1 \wedge c = c_1 \wedge d = d_1$, which means that $f(x) = g(x), x \in \mathbb{R}$. $\square$