Week 6, Chap. 4 (cont.) – Inverse matrix and Chap. 5 – Sequences and series

1 Direct applications

1.1. Let M, P, Q, X be square matrices, with M invertible. The solution of the equation $XM + P = Q$ is $X = M^{-1}(Q - P)$: true or false?

1.2. Determine the inverse of the following matrices:

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$, with $x, y, z \in \mathbb{R}\backslash\{0\}$.

1.3. Prove that the inverse of $\begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{bmatrix}$.

2 Definitions and proofs

2.1. Let A be a matrix $n \times n$. Show that if A is invertible, then $|A| \neq 0$.

2.2. Prove that the inverse of a matrix, whenever exists, is unique.

2.3. Let A and B be invertible matrices and $\lambda \in \mathbb{R}\backslash\{0\}$. Show that the following properties hold:
a) $(A^{-1})^{-1} = A$;
b) $(AB)^{-1} = B^{-1}A^{-1}$;
c) $(\lambda A)^{-1} = \dfrac{1}{\lambda}A^{-1}$.

3 Problems and modelling

3.1. Show that the vectors $\vec{a} = (1, 1, 0)$, $\vec{b} = (0, 1, 1)$ and $\vec{c} = (1, 0, 1)$ are linearly independent:
a) using the definition of linear independence;
b) studying the rank of a matrix;
c) computing the determinant of a matrix;
d) determining if a matrix is invertible.

3.2. Recall problem 3.1 of week 2/3: show that the matrix $R(-\theta)$ is the inverse of $R(\theta)$. Discuss this result from a geometrical point of view.

3.3. Recall problem 3.1 of week 5: write the respective system of equations as $A\vec{x} = \vec{b}$ and determine the solutions $\vec{x}$ by the computation of A^{-1}.

4 Additional exercises

4.1. Let A be a symmetric and invertible matrix with dimension n such that $X'A = B + A$. Then:

a) $X = B'A^{-1} + I$ b) $X = A^{-1}B' + I$ c) $X = \left(\frac{B+A}{A}\right)'$

d) None of the above.

4.2. Consider A, B and C square matrices with dimension n. Knowing that: $|A+B| = 3$, $|C| = 2$ and $(AX+BX)' = C$, obtain X (depending on A, B and C) and compute its determinant.

4.3. Consider the matrix $A = \begin{bmatrix} 1 & x & 0 & 1 \\ x & 1 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 1 & x \end{bmatrix}$.

a) Show that its determinant is $-\left(1-x^2\right)^2$.
b) Find the values of x for which the matrix A does not have an inverse.

4.4. Book:

16.6: 2, 6;
16.7: 2, 5.

5 Direct application

5.1. Book:

10.4: 2 - 4.

5.2. Determine if the following series are convergent. If that is the case, find their values:

a) $\sum\limits_{n\geq 0} \left(\frac{1}{2}\right)^n$ b) $\sum\limits_{n\geq 1} 3^n$ c) $\sum\limits_{n\geq 0} \left(\frac{2}{3}\right)^{n+2}$ d) $\sum\limits_{n\geq 3} \left(\frac{1}{4}\right)^{2n}$ e) $\sum\limits_{n\geq 2} 5^{-n}$.

6 Definitions and proofs

6.1. Define:
a) Function
b) Real-valued function
c) Sequence
d) Series.

6.2. Prove that $\sum\limits_{\ell=0}^{n-1} ak^{\ell} = a\dfrac{k^n - 1}{k-1}$, where $a \in \mathbb{R}$, $k \in \mathbb{R}\backslash\{1\}$, and $n \in \mathbb{N}$.

7 Problems and modelling

7.1. Determine the values of $x \in \mathbb{R}$ for which the following series converge and compute their values:

a) $\sum\limits_{n=1}^{\infty} \left(1-x^2\right)^n$ b) $4x^2 + 16x^4 + 64x^6 + ...$

7.2. Use the theory of the geometric series to write the following numbers as irreducible fractions:

a) $0,999...$ b) $1,666...$ c) $0,1212...$

7.3. Book:

10.4: 6, 7.

8 Additional exercises

8.1. Consider the series $\sum\limits_{n=2}^{\infty} \dfrac{an^2+n}{n^2-1}$, with $a \in \mathbb{R}$. Find the correct answer:

a) if $a \neq 0$, then the series diverges b) if $a \neq 0$, then the series converges
c) the series is convergent, $\forall a \in \mathbb{R}$ d) the series is convergent for $a = 1$.

8.2. Compute $\sum\limits_{n=0}^{\infty} \left[(-\frac{1}{2})^n + (\frac{1}{2})^n\right]$.

8.3. Find the values of $x \in \mathbb{R}$ for which the following series converge and compute their values:

a) $\sum\limits_{n\geq 0} (3x-4)^n$ b) $\sum\limits_{n\geq 0} \left(\dfrac{x-1}{x+1}\right)^n$ c) $\sum\limits_{n\geq 0} \dfrac{2^n}{(x+1)^{2n}}$

d) $\sum\limits_{n=0}^{\infty} \left(\dfrac{x-3}{2}\right)^n$ e) $\sum\limits_{n=0}^{\infty} (1-|x|)^n$ f) $\sum\limits_{n=1}^{\infty} \dfrac{8^n}{(x+1)^{3n}}$ g) $\sum\limits_{n=1}^{\infty} \dfrac{x^{2n}}{2^{n-1}}$.

8.4. Book:

10.4: 5, 8.