Mathematics I

Week 4: Chap. 3 – Systems of linear equations $k \times n$

1 Direct applications

1.1. Consider the following equations/systems: $i)$ $x+3=1$ $\quad ii)$ $x^2=4$ $\quad iii)$ $\begin{cases} x+3=1 \\ x^2=4 \end{cases}$.

a) Identify the non-linear equation.
b) Solve these equations analytically.
c) Solve these equations graphically.

1.2. Determine the set of solutions of the equation $x+y=1$ and classify it.

1.3. Solve and classify the system of equations: $\begin{cases} x+y=1 \\ x+y=-1 \end{cases}$. Verify the result graphically.

1.4. Discuss the existence of solutions for the following systems, finding — whenever possible — the number of degrees of freedom and the solutions:

a) $\begin{cases} -2x-3y+z=3 \\ 4x+6y-2z=1 \end{cases}$ $\quad$ b) $\begin{cases} x-y+2z+w=1 \\ 2x+y-z+3w=3 \\ x+5y-8z+w=1 \\ 4x+5y-7z+7w=7 \end{cases}$

c) $\begin{cases} x-y+z=0 \\ x+2y-z=0 \\ 2x+y+3z=0 \end{cases}$ $\quad$ d) $\begin{cases} x+y+z+w=0 \\ x+3y+2z+4w=0 \\ 2x+y-w=0 \end{cases}$.

1.5. Determine, for the system $\begin{cases} y+az=0 \\ x+by=0 \\ by+az=1 \end{cases}$, depending on the real parameters a and b:

a) the number of independent equations;
b) the number of useless equations;
c) the number of incompatible equations.

1.6. Classify the system of equations depending on the parameter $a \in \mathbb{R}$: $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ x-y-z=a \end{cases}$.

2 Definitions and proofs

2.1. Define equation, system of equations and degree of freedom of a system of equations.

2.2. Let $a \in \mathbb{R}\backslash\{0\}$ and $b, c \in \mathbb{R}$ be constants. Show that $ax+b=c \Leftrightarrow x=\frac{1}{a}(c-b)$.

2.3. Consider three vectors with dimension 4: $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$.
a) Show that the vector equation $\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\lambda_3\vec{v}_3=\vec{0}$, with variables λ_i, correspond to a system of 4 linear equations with 3 variables.
b) Is this system possible? Why?
c) Generalise the previous results for ℓ vectors with dimension p.

2.4. Say, justifying, is the statements below arevtrue or false:

a) A linear system of equations with the same number of equations and of variables has a unique solution.

b) A linear system of equations with the same number of equations and of variables has at least one solution.

c) A linear system of equations with more equations than variables can have an infinite number of solutions.

d) A linear system of equations with less equations than variables can have no solutions.

3 Problems and modelling

3.1. An auto factory uses 3 different types of steel for the production of each of its 3 car models: A, B and C. Each model needs the following amount of steel (in tons):

Steel type \ Car	Model A	Model B	Model C
1	2	3	4
2	1	1	2
3	3	2	1

Find the amount of cars that can be produced using 29, 13 e 16 tons of type 1, 2 and 3 steel, respectively.

3.2. Imagine a region with a closed economy depending on 3 industries: telecom services, electric power and fuel power. The yearly production of these industries is such that:

1. To produce 10 units of telecom services, the telecom industry spends 3 units of its own production, 3 units of electricity and 3 units of fuel.

2. To produce 10 units of electricity, the electric power industry spends 3 units of its own production, 4 units of telecom services and 5 units of fuel.

3. To produce 10 units of fuel, the fuel power industry spends 2 units of its own production, 3 units of telecom services and 6 units of electricity.

Knowing that it is a closed economy, where the production of each industry is the same as the total of what it spends, determine the production of each of the 3 industries.

3.3. Consider: $\begin{cases} x+2y-\alpha z=1 \\ 2x-\ \ y-\ \ z=\beta \\ 9x-2y+\ \ z=-1 \end{cases}$, com $\alpha,\beta\in\mathbb{R}$.

a) Classify the system depending on α and β.
b) Solve it for $\alpha=\beta=0$.
c) Show that the distance between the solution in *b)* and the vector $\left(-\frac{24}{25},-\frac{38}{25},-\frac{2}{5}\right)$ is $\sqrt{5}$.

4 Additional exercises

4.1. Classify the systems with respect to the parameters a and b:

a) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-2y+az=2 \end{cases}$ b) $\begin{cases} x+y+z=1 \\ x-y+2z=a \\ 2x+bz=2 \end{cases}$.

4.2. Let $A\vec{x}=\vec{b}$ a system with 4 equations and 5 variables. Knowing that it has 2 d.o.f., find the rank of the matrix A:

a) 2 b) 3 c) 4 d) 1

4.3. Consider $A=\begin{bmatrix} 1 & 1 & a & 1 \\ 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 3 & 1 & b \end{bmatrix}$, with $a,\ b\in\mathbb{R}$ and $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ with $x_i\in\mathbb{R}$, $i=1,\ldots,4$.

a) Discuss the rank of the matix A as a function of a and b.
b) Find a and b such that $A\vec{x}=\vec{0}$ has a unique solution.

4.4. Let $\begin{cases} x+\ \ y+\ \ \ \ z=1 \\ x+2cy+2cz=1 \\ 2x+\ \ y+\ \ cz=b \end{cases}$.

Find the correct answer:
a) If $c\neq 1$ and $b\neq 1$ the system has infinite solutions
b) If $c=1$ or $c=\frac{1}{2}$, $\forall\ b\in\mathbb{R}$ the system has a unique solution.
c) If $c=1$ and $b\neq 2$ the system has no solutions.
d) If $c\neq\frac{1}{2}$, $\forall\ b\in\mathbb{R}$ the system has infinite solutions.

4.5. Book:

15.1: 1, 3, 5 e 6;
15.6: 1 a 4.