# Week 5: Chap. 4 – Systems of linear equations $n \times n$

## 1 Direct application

**1.1.** Use a determinant to compute the area of the paralelogram defined by the vectors:
a) $\vec{u} = (3, 0)$ and $\vec{v} = (2, 6)$;
b) $\vec{u} = (\alpha, \alpha)$ and $\vec{v} = (\beta, \beta)$, with $\alpha, \beta \in \mathbb{R}$.

**1.2.** Use a determinant to compute the volume of the rectangular prism definined bu the vectors:
a) $\vec{u} = (3, 0, 0)$, $\vec{v} = (2, 6, 0)$ and $\vec{w} = (0, 0, 2)$;
b) $\vec{u} = (1, 0, 0)$, $\vec{v} = (0, 1, 0)$ and $\vec{w} = (0, 0, 1)$;
c) $\vec{u} = (2, 0, 0)$, $\vec{v} = (0, 2, 0)$ and $\vec{w} = (0, 0, 2)$;
d) $\vec{u} = (1, 0, 0, 0)$, $\vec{v} = (0, 1, 0, 0)$, $\vec{w} = (0, 0, 1, 0)$ and $\vec{s} = (0, 0, 1, 0)$.

**1.3.** Use Cramer's rule to solve the following systems os equations, and then check the solutions obtained:

$$\text{a)} \begin{cases} x_1 - x_2 + x_3 = 2 \\ x_1 + x_2 - x_3 = 0 \\ -x_1 - x_2 - x_3 = -6 \end{cases} \qquad \text{b)} \begin{cases} x_1 - x_2 &= 0 \\ x_1 + 3x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \end{cases} \qquad \text{c)} \begin{cases} x + 3y - 2z = 1 \\ 3x - 2y + 5z = 14 \\ 2x - 5y + 3z = 1 \end{cases}.$$

**1.4.** Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, such that $|A| = k$ with $k \in \mathbb{R}$. Find the value of:

$$\text{a)} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}; \quad \text{b)} \begin{vmatrix} 2a_{11} & a_{12} & a_{13} \\ 2a_{21} & a_{22} & a_{23} \\ 2a_{31} & a_{32} & a_{33} \end{vmatrix}; \quad \text{c)} \left| 3 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right|;$$

$$\text{d)} \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}; \quad \text{e)} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \quad \text{f)} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{vmatrix}.$$

**1.5.** The value of $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & x & y \\ 2 & 1 & y & x \end{vmatrix}$ is equal to:

a) $-x^2 + y^2 + x + y$ b) $x^2 + y^2 + x - y$
c) $x^2 - y^2 - x - y$ d) None of the above.

**1.6.** Book:
**16.5:** 2.

## 2 Definitions and proofs

**2.1.** Show that if a square matrix has a zero row or column, then its determinant is zero.

**2.2.** Let $M$ be an upper triangular matrix of order 4.
a) Prove that its determinant is the product of the coefficients on the diagonal.
b) Show that the previous result is also true for a lower triangular matrixof order 4.

**2.3.** A square matrix $M$ is said to be *orthogonal* if $M'M = I$, where $I$ is the identity matrix. Prove that:
a) The product of two orthogonal matrices $n \times n$ is again an orthogonal matrix.
b) The determinant of an orthogonal matrix is always $+1$ or $-1$.

**2.4.** Without computing the determinants, show that: $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$.

## 3 Problems and modelling

**3.1.** A company in the food business produces daily wheat flour, rye flour and corn flour. The total production is $k$ tons per day. Knowing that the total daily production of wheat and rye flours is three times the corn flour production, and that the total daily production of wheat and corn flours is twice the rye flour production, use Cramer's rule to find the wheat flour daily production.

**3.2.** Consider: $\begin{cases} x+z &= a \\ y+az &= 0 \\ x+(a+1)\,z &= a+b \end{cases}$, with $a, b \in \mathbb{R}$.
a) Use Cramer's rule to find $a$ and $b$ so that this system has a unique solution. What can we say about the system for other values of $a$ and $b$?
b) Determine the solutions of the system when $a = 1$ and $b = 1$.

**3.3.** Book:
**16.1:** 6 e 7.

## 4 Additional exercises

**4.1.** Let $A$ and $B$ square matrices of order $n$ such that $|A| = k$ and $|B| = q$, with $kq \neq 0$. The determinant of $C = qkAB$ is:

a) $(qk)^n$ b) $(qk)^{n+1}$ c) $(qk)^2$ d) None of the above.

**4.2.** Consider: $\begin{bmatrix} 1 & k & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $k \in \mathbb{R}$.
a) Determine $k$ such that the distance between columns 1 and 2 of the matrix containing the systems coefficients is $2\sqrt{2}$.
b) Consider $k = 0$ and check the triangular inequality for the distances between the columns of the matrix.
c) Consider $k = 1$ and use Cramers's rule to find $y$.

**4.3.** Book:
**16.3:** 1 e 2;
**16.4:** 1 a 8;
**16.5:** 1.