Week 7: Chap. 5 – Real functions, and Chap. 6 – Variations

1 Direct applications

1.1. Book:
6.5: 1, 4.

1.2. Sketch the graph of the following functions:

a) $-x^2$ b) $-\sqrt{x}$ c) e^x d) $\ln x$ e) $\dfrac{1}{x}$ f) $\sin x$ g) $\cos x$ h) $\tan x$

i) $ax+b$ with $a, b \in \mathbb{R}$ j) $|x+5|$ k) $\ln(x-5)$ ℓ) an odd function.

1.3. Compute the derivative with respect to x of the functions in questions *a)* to *i)* in 1.2.

1.4. For which values of a and b the function $f(x) = \begin{cases} ax-2 & \text{se } x \leq 1 \\ b-2x^2 & \text{se } x > 1 \end{cases}$ is continuous?

1.5. Let $f(x) = e^x$, $g(x) = x^n$ with $n \in \mathbb{Z}$, and $h(x) = \sin x$. Compute:

$$a)\frac{d}{dx}\left[f(x)+g(x)+h(x)\right] \quad b)\frac{d}{dx}\left[5f(x)+2g(x)\right] \quad c)\frac{d}{dx}\left[g(x)h(x)\right]$$
$$d)\frac{d}{dx}\left[f(x)g(x)h(x)\right] \quad e)\frac{d}{dx}\left[\frac{h(x)}{f(x)}\right] \quad f)\frac{d}{dx}\left[\frac{g(x)h(x)}{f(x)}\right].$$

1.6. Let $f(x) = \sqrt{x}$.

a) Find the domain of f and discuss its continuity and differentiability.

b) Compute: $\dfrac{df(x)}{dx}$, $\dfrac{d^2f(x)}{dx^2}$ and $\dfrac{d^3f(x)}{dx^3}$.

2 Definitions and proofs

2.1. Prove by the definition that: $\lim\limits_{x \to 2} 3x+1 = 7$.

2.2. Consider the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$. Show that if f and g are continuous in $a \in \mathbb{R}$, then $(f+g)$ is also continuous in a.

2.3. Let $f(x) = x^2$. Prove by the definition that: $\dfrac{df(x)}{dx} = 2x$.

2.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Show that $\dfrac{f(a)-f(x)}{a-x} = \dfrac{f(x+h)-f(x)}{h}$, with $h = a-x$.

2.5. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions and $k \in \mathbb{R}$. Show that:

a) $\dfrac{d}{dx}\left[f(x)+g(x)\right] = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$.

b) $\dfrac{d}{dx}\left[kf(x)\right] = k\dfrac{df(x)}{dx}$.

3 Problems and modelling

3.1. The stock price for the following companies is given with respect to time t by:

- Company A: $2t^2 + 4t$
- Company B: $3t^2 + t$
- Company C: $\frac{2t}{t^2+1}$.

a) At $t = 1$ which company has the fastest growing stock price?
b) In what period of time the stock price of C is growing?

3.2. Study the domain, continuity and diferentiability of:

a) $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$ b) $g(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{se } x < 0 \\ \ln\left(1 + x^2\right) & \text{se } x \geq 0 \end{cases}$.

3.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Solve the equation: $\dfrac{df(x)}{dx} = f(x)$.

3.4. Book:
7.8: 4;
6.7: 8;
6.9: 9, 10.

4 Additional exercises

4.1. Determine the domain of:

a) $f(x) = \dfrac{1}{x+3}$ b) $g(x) = \dfrac{x}{x^2+1}$ c) $h(x) = \ln(3 - 2x)$

d) $i(x) = \sqrt{x^2 - 25}$ e) $j(x) = \dfrac{1}{\sqrt{x^2 - 4}}$ f) $k(x) = \ln(\ln x)$

g) $l(x) = \dfrac{1}{\ln(1 - |x - 1|)}$ h) $m(x) = \dfrac{\ln(4 - x^2)}{\sqrt{e^x - 1}}$.

4.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a two times differentiable function. Solve the equation: $\dfrac{d^2 f(x)}{dx^2} = -f(x)$.

4.3. Book:
7.8: 2, 3, 5;
7.9: 1 to 3;
6.5: 5;
6.7: 6, 7;
6.9: 1, 3, 7.