Week 8: Chap. 6 – Diferentials, Composite functions, De l'Hôpital rule

1 Direct applications

1.1. Let $f(x) = e^x$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$. Determine the domain and the range of:

a) $f \circ g$ b) $f \circ h$ c) $h \circ f$ d) $h \circ f \circ g$ e) $f \circ h \circ f \circ g$

1.2. Compute the diferential of the following functions with respect to their variable:

a) $x^5 + 2x^4 + 1$ b) $-\sqrt{u}$ c) e^y d) $\ln z$ e) $\dfrac{1}{x}$ f) $\sin u$ g) $\dfrac{\sin x}{\cos x}$.

1.3. Differentiate the following functions with respect to x:

a) $\left(5x^{70} + 3x + 1\right)^2$ b) $\left(5x^2 + 3x + 1\right)^{70}$ c) $\cos\left(3x^5 - x\right)$ d) $e^{-\frac{x}{2}}$

e) $\sqrt{x-3}$ f) $\dfrac{1}{\ln x}$ g) $e^{\sin x}$ h) $x + \sqrt{x^2 - 1}$

i) $\ln(\sin x)$ j) $\ln\left(x^2 + 1\right)$ k) $\ln^4\left(\sqrt{1 - x^2}\right)$ ℓ) $e^{-\cos\left(\sqrt{x^4+x^2+1}\right)}$

1.4. Find the limits:

a) $\displaystyle\lim_{x\to\frac{\pi}{3}} \frac{\sin 3x}{1 - 2\cos x}$ b) $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{e^{\sin x} - e^{\cos x}}{\sin x - \cos x}$ c) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{(2x-1)^2}{e^{2x-1} - 4x^2}$

d) $\displaystyle\lim_{x\to+\infty} x \ln\left(1 + \frac{1}{x}\right)$ e) $\displaystyle\lim_{x\to-\infty} x e^{-x^2}$ f) $\displaystyle\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

2 Problems and modelling

2.1. Three plastic companies have the following production costs, depending on the oil price p:

- Company 1: $5p^3 + 2p + 1$
- Company 2: $2p^{3/2} + p$
- Company 3: $\sqrt{p} + \frac{1}{p}$.

a) Determine for each company the average rate of change for the production cost when the oil changes price from 1 €/ℓ to 4 €/ℓ.
b) Determine for each company the instantaneous rate of change of the production cost when the oil price is 1 €/ℓ.
c) Knowing that during a brief period of crisis $t \in [0, 2]$ the oil price was $p(t) = e^t$, determine which company had a faster growing production cost at $t = 1$.

2.2. Consider the function $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ e^{-kx} & \text{if } x \geq 0 \end{cases}$, with $k > 0$.

a) Find the domain of f and sketch its graph.
b) Discuss the continuity of f on its domain.
c) Discuss the differentiability of f on its domain.
d) Consider the function $g(x) = \sqrt{x}$. Discuss the continuity and differentiability of $g \circ f$ and compute its derivative where possible.

2.3. Let $h(x) = f(x \ln x)$ be differentiable on $\mathbb{R}$. Knowing that $f(0) = \sqrt{3}$ and $f'(0) = 2$, find the equation of the tangent line to the graph of h at $x = 1$.

3 Additional exercises

3.1. Differentiate the following functions with respect to x:

a) $\left(\dfrac{x-1}{x+2}\right)^2$ b) $\left(\dfrac{x^2-1}{2x}\right)^3$ c) $\sqrt{e^x+1}$ d) $e^{-\sqrt{x}}$

e) $e^{x^3} \ln\left(x^2\right)$ f) $\dfrac{3}{\sqrt{x}}$ g) $\sqrt[3]{\dfrac{3-x}{x-1}}$ h) e^{x^2}

i) $\ln\left(e^{3x}+x^2\right)$ j) $e^x \ln x$ k) $\sin(2x+1)$ ℓ) xe^x

m) $\cos x + x\cos^2\left(x^2\right)$ n) $\sin x \cos x$ o) $\tan\left(x^2+1\right)$ p) $\ln\dfrac{1+x}{1-x}$

3.2. Compute the limits:

a) $\displaystyle\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$ b) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin x}$

3.3. Find $L = \lim\limits_{x\to 1} \frac{2x^\alpha - 2\alpha(x-1)-2}{3x^2-6x+3}$:

a) $L = -\alpha - 3$ b) $L = 0$ c) $L = \frac{\alpha^2-\alpha}{3}$ d) L does not exist

3.4. Find $\displaystyle\lim_{x\to+\infty} \frac{\sin(5/x)}{2/x}$?

a) $\frac{5}{2}$ b) 0 c) $-\frac{5}{2}$ d) $\frac{2}{5}$.

3.5. Let f and g be differentiable functions on $\mathbb{R}$ such that $h(x) = f[g(x)]$. Knowing that $f(-1) = 2$, $f'(-1) = 1/3$, $g(3) = -1$, and $g'(3) = -4$, find the equation of the line tangent to the graph of h at $x = 3$:

a) $y = -\frac{4}{3}x + 2$ b) $y = -\frac{4}{3}x + 6$ c) $y = -4x + 2$ d) $y = -x + 5$

3.6. Book:
6.2: 5, 7.