Maths I

Week 13: Chap. 9 – Integrals and areas (Part II)

1 Direct applications

1.1. Study the convergence of the following improper integrals, and find their values whenever possible:

a) $\int_0^{+\infty} e^{-x}dx$ b) $\int_{-\infty}^{-1} \frac{1}{x}dx$ c) $\int_0^1 \frac{1}{\sqrt{x}}dx$ d) $\int_{-\infty}^{+\infty} u^3du$ e) $\int_{-\infty}^{+\infty} e^{-x}dx.$

1.2. Compute the following anti-derivatives by parts:

a) $\int xe^x dx$ b) $\int x^2 \ln x dx$ c) $\int t \sin t dt$ d) $\int x^2 \sin x dx$ e) $\int e^x \cos x dx.$

1.3. Compute the following anti-derivatives by substitution:

a) $\int 2x \sin(x^2)dx$ b) $\int e^{x^2} x dx$ c) $\int \frac{x^4}{x^5+7}dx$ d) $\int \frac{x}{1+x^4}dx$ e) $\int \frac{\cos x}{2\sqrt{\sin x}}dx.$

1.4. Determine: a) $\frac{d}{dt}\int_4^t e^{-x^2}dx$ b) $\frac{d}{dx}\int_x^{\alpha} \frac{1}{\sqrt{s^4+1}}ds$, with $\alpha \in \mathbb{R}$.

1.5. Find the area of the following subsets of $\mathbb{R}^2$:

a) $\{(x,y) \in \mathbb{R}^2 \colon y \leq 5, y \geq -5x+5, y \geq \ln x\}$

b) $\{(x,y) \in \mathbb{R}^2 \colon 0 \leq y \leq e^x, x \leq 1\}$

c) $\{(x,y) \in \mathbb{R}^2 \colon x \leq y \leq -x^2+2\}$.

1.6. Compute:

a) $\int_0^{\sin t} x^3dx$ b) $\int_1^e \ln x dx$ c) $\int_0^1 te^{-t^2}dt$ d) $\int_{-\pi}^{\pi} x\cos x dx$ e) $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+\sin^2 x}dx$

f) $\int_0^1 \sqrt{3x+7}dx$ g) $\int_{-3}^4 |x-2|dx$ h) $\int_0^1 3e^{5x+1}dx$ i) $\int_0^{\pi} 2xt^3 \cos t^4 dt$ j) $\int_1^2 \frac{1}{x}dx.$

2 Problems

2.1.

a) Find the area above the graph of $f(x) = x^2 - 4$ and below the x-axis.

b) Find the area between the graphs of $f(x) = x^2+x+1$ and $g(x) = 2x^2+5x+4$, for $x \in [-3,0]$.

2.2. Determine the domain, the intervals of monotony and the local extreme points of:

a) $F(x) = \int_1^x \ln t dt$ b) $H(x) = \int_0^{x^2} e^{-t^2}dt.$

2.3. Consider the function $f(x) = \int_{\pi}^{x^2} e^{-2t} dt$. Write Taylor's formula with order 1 of f around $x = 0$.

2.4. Consider the function $F(x) = \int_0^x t f(t) dt$, where f is continuous and strictly positive in $\mathbb{R}$. Prove that F has a local minimum at $x = 0$.

2.5. Consider the function $f(x) = \begin{cases} a - x & \text{if } x < 0 \\ \frac{1}{b+x} & \text{if } x \geq 0 \end{cases}$, with $b > 0$.

a) Determine the values of a and b for which f is continuous.

b) Take $a = b = 1$ and consider the function $F(x) = \int_{-1}^{x} f(t) dt$. Find $F(1)$. Moreover, show that F is invertible on $(0, +\infty)$.

2.6. Let $f(x) = \int_1^{x^2+1} \left(\frac{1+t}{t} \right) dt$.

a) Find $f(-1)$.

b) Determine the equation of the tangent line to the graph of f at $x = -1$.

2.7. Consider the function with domain D_f: $f(x) = \begin{cases} -x^2 - x + 1 & \text{if } x < 0 \\ e^{2x} & \text{if } x \geq 0 \end{cases}$.

a) For which values of $x \in D_f$ the function f is differentiable? Find $f'(x)$.

b) Determine $G(x) = \int_{-1}^{x} f(t) dt$, defined in $[-1, \infty)$.

2.8. Determine the function f, twice differentiable on $\mathbb{R}$, that satisfies: $f''(x) = 2\cos x + xe^x$, $f'(0) = 2$, $f(0) = 1$

2.9. Without using anti-derivatives, compute:

a) $\lim\limits_{x \to 0} \dfrac{\int_0^x \ln\left(t^2 + 1\right) dt}{x^3}$ b) $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} \cos t dt}{x^2}$.

3 Additional exercises

3.1. Determine an anti-derivative of the following functions, in their respective domains:

a) $x^2 e^x$ b) $x\sqrt{x+1}$ c) $x^3\sqrt{1+x^2}$ d) $2x \cos x$ e) $\sin^2 x$

f) $\ln(2x - 1)$ g) $x^2 \ln x$ h) $\arctan x$ i) $\ln^2 x$ j) $e^x \cos x$.

3.2. Determine, by substitution, an anti-derivative of:

a) $\dfrac{x}{1+x^2}$ b) $\sqrt{1 - \sin^2 x}$

c) $\dfrac{e^{\frac{x}{4}}}{1 + e^{\frac{x}{10}}}$, with $x = 20 \ln t$ $(t > 0)$ d) $\dfrac{\cos x}{\sin^6 x}$, with $x = \arcsin t$.

3.3. Study the convergence of the improper integrals and find their values whenever possible:

a) $\displaystyle\int_0^{+\infty} xe^{-x^2}dx$ b) $\displaystyle\int_0^{+\infty} \cos x dx$ c) $\displaystyle\int_0^{+\infty} \frac{\arctan x}{1+x^2}dx$

d) $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+1}dx$ e) $\displaystyle\int_0^{3} \frac{1}{x-3}dx$ f) $\displaystyle\int_0^{2} \frac{2}{\sqrt{4-x^2}}dx.$

3.4. Find the area aobve the graph of $f(x) = \ln x$, for $x \in [0,1]$, and below the line $y = 0$.

3.5. Book:

9.3: 4 to 6;
9.5: 2, 3;
9.6: 3;
9.7: 1, 4, 12.