Week 12: Chap. 9 – Integrals and areas (Part I)

1 Direct applications

1.1. Compute the following anti-derivatives:

a) $\int x^2 dx$ b) $\int \sqrt{x} dx$ c) $\int e^x dx$ d) $\int \cos y dy$ e) $\int \frac{x^5}{5} dx$ f) $\int \frac{1}{2\sqrt{x}} dx$

g) $\int \frac{1}{2} dx$ h) $\int x^4 dt$ i) $\int (\sin u + x^2) dx$ j) $\int (\sin u + x^2) du$ k) $\int e^{7u} dx$ ℓ) $\int \frac{1}{2} dt$.

1.2. Compute the anti-derivative $F(x) = \int f(x)dx$:

a) such that $F(2) = 0$, for $f(x) = x^4$;
b) such that $F(0) = 1$, for $f(x) = e^x$;
c) such that $F(1) = \pi$, for $f(x) = x^{-1}$;
d) such that $F(0) = e$, for $f(x) = x^3 - 4x^2 + 4x + 12$;
e) such that $F(1) = 0$, for $f(x) = (1 - x^2)^{-\frac{1}{2}}$.

1.3. Compute the following integrals:

a) $\int_0^2 x^3 dx$ b) $\int_1^0 (-\sqrt{x}) dx$ c) $\int_0^{\ln 1} e^{-t} dt$ d) $\int_{-\pi}^{\pi} \cos y dy$ e) $\int_0^1 \frac{1}{1+x^2} dx$

f) $\int_{-1}^1 (6x^5 + \frac{1}{3}x^2 - 2x + 7) dx$ g) $\int_2^3 (\sin u + x^{\frac{1}{3}}) dx$ h) $\int_e^{7e} e^{7u} dx$ i) $\int_a^b 1 dt$.

1.4. Find the area between the graph of f and the x-axis for:

a) $f(x) = x^2$ and $x \in [0, 2]$
b) $f(x) = -x^2$ and $x \in [0, 2]$
c) $f(t) = e^{-t}$ and $t \in [1, 5]$
d) $f(x) = -\sqrt{\sqrt{x}}$ and $x \in [0, 1]$
e) $f(x) = \frac{-x^4 - 2x^2}{x}$ and $x \in [-1, 1]$

2 Definitions and proofs

2.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}$, and $a, b, \lambda \in \mathbb{R}$ constants. Show that:

a) $\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$.

b) $\int_a^b f(x) dx = -\int_b^a f(x) dx$.

c) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, with $a \le c \le b$.

2.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an odd continuous function, and $k \in \mathbb{R}$.

a) Prove that $\displaystyle\int_{-k}^{k} f(x)dx = 0$.

b) Interpret geometrically the previous result.

2.3. Let $a, b \in \mathbb{R}$ such that $a < b$, and $d(a, b)$ the distance between these two points.

a) Show that $d(a, b) = \displaystyle\int_{a}^{b} dx$.

b) Interpret geometrically the previous result.

3 Problems and modelling

3.1. An oil well has an extraction rate (measured in barrels by unit of time) that varies with time t according to: $10e^{-2t}$.

a) What is the amount of oil extracted from the well at time $t = 50$?

b) Solve the same problem for the rate 2^{-t}.

3.2. Let $f(x) = \sin x$. Find the area between the graph of f and the x-axis for:

a) $x \in [0, \frac{\pi}{2}]$

b) $x \in [-\frac{\pi}{2}, 0]$

c) $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

d) Discuss the results from the geometrical point of view.

e) Discuss the results from the point of view of exercise 2.2.

3.3. Let $f(x) = x^3 - 4x^2 + 4x$. Compute the area between the graph of f and the x-axis for $x \in [-1, 2]$.

4 Additional exercises

4.1. Book:

9.1: 1 to 9;

9.2: 1 to 6, 8.