Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM

January 4, 2011

2 hours, No consultation, No calculators

1. (1.5) Find k that minimizes the distance between $\vec{u} = (-1, 1, k, 0)$ and $\vec{v} = (2, 0, 0, -7)$.

2. (1.5) For which values of $x \in \mathbb{R}$ the series $\sum_{n=0}^{+\infty} 7(\cos x)^n$ converges?

3. (1.5) Find $\lim_{x \to 0^+} x^{5x}$.

4. (1.5) Compute the following determinant:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \end{vmatrix},$$

   where $\alpha, \beta, \gamma \in \mathbb{R}$.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:

$$\begin{cases} x + 2y + z = 0 \\ -x - y + \alpha z = 3 \\ -2x - 3y + z = \beta \end{cases}$$

   a. (2.5) Classify this system depending on α and β.
   b. (0.5) Solve this system for $\alpha = 2$ and $\beta = 3$.

6. (0.5) Define rank of a matrix.

7. (0.5) Consider the vectors $\vec{a} = (0, 2, 0)$, $\vec{b} = (2, x, 0)$ and $\vec{c} = (0, 0, y)$, where $x, y \in \mathbb{R}$. Find for which values of x and y the vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

8. (1.5) Let A be an invertible matrix. Show that $(A^{-1})^{-1} = A$.

9. Consider the function $f(x) = xe^x + 3$.
   a. (0.5) Find the domain of f and discuss its continuity.
   b. (1.0) Determine the stationary points of f.
   c. (1.0) Determine the extreme points of f using the second derivative.
   d. (1.0) Discuss if the above extreme points are global.
   e. (1.0) Write the quadratic approximation of f around $x = 0$.
   f. (0.5) Find the intervals where f is invertible.
   g. (1.0) Determine the derivative of the inverse function f^{-1} at 3.

10. (1.5) Consider $f(x) = -e^{-x}$ and $g(x) = e^{-x}$. Compute the area between the graphs of f and g, with $x > 0$.

11. (1.0) Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain, and $p \in \mathbb{R}$. Prove that $\text{El}_x \left[f(x)^p \right] = p \ \text{El}_x \left[f(x) \right]$.