Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM

January 24, 2011

2 hours, No consultation, No calculators

1. (1.5) Let $\vec{u} = (1, 2, 0, k)$ and $\vec{v} = (3, -1, -1, -1)$. Find k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

2. (1.5) Let $f: \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain. Compute $El_x \dfrac{1}{f(x)}$.

3. (1.5) For each value of $\beta \in \mathbb{R}$ find

$$\lim_{x \to 1} \frac{x^3 - 3\beta x + 3\beta - 1}{(x-1)^2}.$$

4. (1.5) Decide if the function $f(x) = \sqrt{x + \alpha}$, with $\alpha \in \mathbb{R}$, is linear or non-linear.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:

$$\begin{cases} x + y + z = 1 \\ 2x + 5z = 1 \\ x - y + \alpha z = \beta \end{cases}$$

   a. (2.0) Classify this system depending on α and β.
   b. (1.0) Solve this system for $\alpha = 0$ and $\beta = 0$ using Cramer's rule.

6. (1.0) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

7. (1.0) Let A, B, C, X be $n \times n$ matrices and let I be the $n \times n$ identity matrix. Assuming that C and $A - B$ are invertible, solve the equation $AXC = BXC + I$ with respect to X.

8. (1.5) Consider the series
$$\sum_{n=0}^{+\infty} \left(\frac{3x+2}{3} \right)^n .$$
Discuss for which values of x the series converges, and compute its sum whenever possible.

9. Consider the function $g(x) = e^{(x^2+1)}$.
   a. (0.5) Find the domain of g and discuss its continuity.
   b. (1.0) Determine the stationary points of g.
   c. (1.0) Determine the extreme points of g using the second derivative.
   d. (0.5) Study the concavity of g.
   e. (0.5) Discuss if the above extreme points are global.
   f. (1.5) Compute $\displaystyle\int_0^1 xg(x)\,dx$.

10. Recall that
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$
   a. (1.5) Estimate the value of $\arctan(0.1)$.
   b. (1.0) Give an upper bound for the approximation error.