Week 10: Chap. 7 – Polynomial approximations, Intermediate value theorem and mean value theorem

1 Direct application

1.1. Let $f(x) = \ln x$.
a) Find the linear approximaton of f around $x = 1$.
b) Find the quadratic approximation of f around $x = 1$.
c) Sketch the graph of f and compare it with the graphs of the previous approximations.
d) Make an estimate of $\ln(1.1)$.

1.2. The quadractic approximation of $f(x) = (x + 1)^5$ around $x = 1$ is given by:

a) $f(x) \simeq 80x^2 - 80x + 32$ b) $f(x) \simeq -80x^2 + 80x + 32$

c) $f(x) \simeq -80x^2 - 80x - 32$ d) $f(x) \simeq 80x^2 + 80x + 32$

1.3. Let $f(x) = \left(\frac{1}{x} - 1\right)^2$. The Taylor approximation of second degree of f around $x = 1$ is:

a) $x - 1 + (x - 1)^2$ b) $x - 1 - (x - 1)^2$

c) $-(x - 1)^2$ d) $(x - 1)^2$

1.4. Write Taylor's formula of degree n for $f(x) = e^x$ around $x = 1$, with the Lagrange's remainder. Compute the limit of the remainder when $n \to +\infty$.

1.5. Show that the equation $xe^x = \dfrac{1}{2}$ has one unique solution in $]-1, 1[$.

2 Definitions and proofs

2.1. Use the linear approximation to show that around the origin we have: $\sin x \simeq x$.

2.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$.
a) Define increasing function.
b) Prove that if $f'(x) \geq 0$ for $x \in]a, b[$, then f is increasing.

2.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable on $\mathbb{R}$ and let $p(x) = \alpha x^2 + \beta x + \gamma$, with $\alpha, \beta, \gamma \in \mathbb{R}$. Determine the coefficients α, β, γ that satisfy the following conditions for $a \in \mathbb{R}$:

$$\begin{cases} f(a) = p(a) \\ f'(a) = p'(a) \\ f''(a) = p''(a) \end{cases} .$$

3 Problems and modelling

3.1. Estimate the approximate value of $\sin(0.1)$ and its approximation error.

3.2. Let f be implicitly defined by the equation $[f(x)]^3 = x^3 f(x) + x + 1$. Knowing that $f(0) = 1$, find the linear aproximation of $f(x)$ around $x = 0$.

3.3. Consider $f(x) = e^{x-1}$.
a) Write the Taylor formula of degree n of f around 1.
b) Find an upper bound on the remainder for $x = \frac{1}{2}$ and $n = 3$.

3.4. Use the Taylor formula to compute: $\lim_{x \to 0} \frac{\sin x - x}{x^2}$.

3.5. Let $f(x) = \sqrt{x}$. Determine the linear approximation of f around $x = 1$ and use it to get an approximation of $\sqrt{1.1}$.

4 Additional exercises

4.1. Use the Taylor formula to write the polynomial $x^3 - 2x^2 - 5x - 2$ as a sum of powers of $(x + 2)$.

4.2. Let $y = f(x)$ implicitly defined by $xy - x^2 = 2y + x$. The linear approximation of f around 4 is given by:

a) $-5x + 3$ b) $-\frac{1}{2}(x - 24)$ c) $\frac{1}{3}(x + 25)$ d) $x + 3$

4.3. Let $f(x) = (2x - a)^m$, with $m \in \mathbb{N}$. Show that the Taylor approximation of second degree of f around 0 is:

$$(-a)^m + 2m(-a)^{m-1}x + 2m(m-1)(-a)^{m-2}x^2.$$

4.4. Book:
7.4: 1 to 4, 7, 9, 10
7.5: 1, 2, 4, 5
7.6: 1, 2, 4;
7.10: 1, 2;
8.4: 6, 7.