Week 11: Chap. 8 – Extremes and concavities, Chap. 9 – Integrals and areas

1 Direct applications

1.1. Book:

8.6: 4, 5;
8.7: 5, 6.

1.2. Let $f(x) = x^3 - 4x^2 + 4x + 12$.

a) Determine the stationary points of f.
b) Determine the extreme points of f using the second derivative.
c) Find out if the extreme points are local or global.

1.3. Let $f\colon I \to \mathbb{R}$ such that $f(x) = \sin(x^2)$, with $I = [-\sqrt{\pi}, \sqrt{\pi}]$.

a) Determine the stationary points of f.
b) Determine the extreme points of f using the second derivative.
c) Find out if the extreme points are local or global.

1.4. Let $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$.

a) Determine the stationary points of each function.
b) Using the derivatives of order 2 or higher, determine if those points are minima, maxima or inflection points.
c) Determine the concavities of each function.

1.5. Is an inflection point always a stationary point?

2 Definitions and proofs

2.1. State the definition of increasing and decreasing functions.

2.2. State the definition of stationary point.

2.3. Let $f\colon\mathbb{R} \longrightarrow \mathbb{R}$ have a second derivative continuous on I, and a an interior point of I.
a) State the definition of inflection point of f.
b) Prove that if a is an inflection point of f, then $f''(a) = 0$.

3 Problems and modelling

3.1. A faulty freezer operates between -3ºC and +2ºC, and it has an energy consumption that varies with the temperature t as: $t^3 + \frac{3}{2}t^2 - 6t + 10$.

a) Determine the temperatures for which the energy consumption is maximum and minimum.
b) Does the function energy consumption have an inflection point?

3.2. Let $f(x) = \begin{cases} (x+2)^2, & x < -1 \\ |x|, & -1 \leq x \leq +1 \\ e^{-x+1}, & x > +1 \end{cases}$.

a) What is the domain of f?
b) Discuss the continuity and differentiability of f in its domain.
c) Determine the stationaty points of f.
d) Determine the extreme points of f, indicating if local or global.
e) Determine the extreme points of f in $[-4, -1]$.

3.3. Consider $f(x) = x \sin x$.
a) Find the Taylor polynomial of second degree of f around 0.
b) The function f has a unique stationary point in $]-1, 1[$. Determine it.
c) Classify this stationary point using the second derivative.
d) Is there any extreme points of f in $]-1, 1[$?

4 Additional exercises

4.1. Let f be the function and I the interval in exercise 1.3. Show that f has at least two inflection points in I.

4.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $a \in \mathbb{R}$ such that $f'(a) = 0$ and $f''(a) < 0$. Prove that a is a local maximum of f.

4.3. Book:

8.6: 1, 3, 6;
8.7: 2 to 4.

5 Direct applications

5.1. Compute the following anti-derivatives:

a) $\int x^2 dx$ b) $\int \sqrt{x} dx$ c) $\int e^x dx$ d) $\int \cos y dy$ e) $\int \frac{x^5}{5} dx$ f) $\int \frac{1}{2\sqrt{x}} dx$
g) $\int \frac{1}{2} dx$ h) $\int x^4 dt$ i) $\int (\sin u + x^2) dx$ j) $\int (\sin u + x^2) du$ k) $\int e^{7u} dx$ ℓ) $\int \frac{1}{2} dt$.

5.2. Compute the anti-derivative $F(x) = \int f(x) dx$:

a) such that $F(2) = 0$, for $f(x) = x^4$;
b) such that $F(0) = 1$, for $f(x) = e^x$;
c) such that $F(1) = \pi$, for $f(x) = x^{-1}$;
d) such that $F(0) = e$, for $f(x) = x^3 - 4x^2 + 4x + 12$;
e) such that $F(1) = 0$, for $f(x) = (1 - x^2)^{-\frac{1}{2}}$.

5.3. Compute the following integrals:

a) $\int_0^2 x^3 dx$ b) $\int_1^0 (-\sqrt{x})dx$ c) $\int_0^{\ln 1} e^{-t}dt$ d) $\int_{-\pi}^{\pi} \cos y dy$ e) $\int_0^1 \frac{1}{1+x^2}dx$

f) $\int_{-1}^{1} (6x^5 + \frac{1}{3}x^2 - 2x + 7)dx$ g) $\int_2^3 (\sin u + x^{\frac{1}{3}})dx$ h) $\int_e^{7e} e^{7u}dx$ i) $\int_a^b 1dt.$

5.4. Find the area between the graph of f and the x-axis for:
a) $f(x) = x^2$ and $x \in [0, 2]$
b) $f(x) = -x^2$ and $x \in [0, 2]$
c) $f(t) = e^{-t}$ and $t \in [1, 5]$
d) $f(x) = -\sqrt{\sqrt{x}}$ and $x \in [0, 1]$
e) $f(x) = \frac{-x^4 - 2x^2}{x}$ and $x \in [-1, 1]$

6 Definitions and proofs

6.1. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}$, and $a, b, \lambda \in \mathbb{R}$ constants. Show that:

a) $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$

b) $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, with $a \leq c \leq b$.

6.2. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be an odd continuous function, and $k \in \mathbb{R}$.
a) Prove that $\int_{-k}^{k} f(x)dx = 0.$
b) Interpret geometrically the previous result.

6.3. Let $a, b \in \mathbb{R}$ such that $a < b$, and $d(a, b)$ the distance between these two points.
a) Show that $d(a, b) = \int_a^b dx.$
b) Interpret geometrically the previous result.

7 Problems and modelling

7.1. An oil well has an extraction rate (measured in barrels by unit of time) that varies with time t according to: $10e^{-2t}$.

a) What is the amount of oil extracted from the well at time $t = 50$?
b) Solve the same problem for the rate 2^{-t}.

7.2. Let $f(x) = x^3 - 4x^2 + 4x$. Compute the area between the graph of f and the x-axis for $x \in [-1, 2]$.

8 Additional exercises

8.1. Book:

9.1: 1 to 9;
9.2: 1 to 6, 8.