

Week 12: Chap. 9 – Integrals and areas

1 Direct applications

1.1. Study the convergence of the following improper integrals, and find their values whenever possible:

$$\text{a) } \int_0^{+\infty} e^{-x} dx \quad \text{b) } \int_{-\infty}^{-1} \frac{1}{x} dx \quad \text{c) } \int_0^1 \frac{1}{\sqrt{x}} dx \quad \text{d) } \int_{-\infty}^{+\infty} u^3 du \quad \text{e) } \int_{-\infty}^{+\infty} e^{-x} dx.$$

1.2. Compute the following anti-derivatives by parts:

$$\text{a) } \int x e^x dx \quad \text{b) } \int x^2 \ln x dx \quad \text{c) } \int t \sin t dt \quad \text{d) } \int x^2 \sin x dx \quad \text{e) } \int e^x \cos x dx.$$

1.3. Compute the following anti-derivatives by substitution:

$$\text{a) } \int 2x \sin(x^2) dx \quad \text{b) } \int e^{x^2} x dx \quad \text{c) } \int \frac{x^4}{x^5 + 7} dx \quad \text{d) } \int \frac{x}{1 + x^4} dx \quad \text{e) } \int \frac{\cos x}{2\sqrt{\sin x}} dx.$$

1.4. Determine: a) $\frac{d}{dt} \int_4^t e^{-x^2} dx$ b) $\frac{d}{dx} \int_x^\alpha \frac{1}{\sqrt{s^4 + 1}} ds$, with $\alpha \in \mathbb{R}$.

1.5. Find the area of the following subsets of \mathbb{R}^2 :

- a) $\{(x, y) \in \mathbb{R}^2 : y \leq 5, y \geq -5x + 5, y \geq \ln x\}$
 b) $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq e^x, x \leq 1\}$
 c) $\{(x, y) \in \mathbb{R}^2 : x \leq y \leq -x^2 + 2\}$.

1.6. Compute:

$$\begin{aligned} \text{a) } \int_0^{\sin t} x^3 dx & \quad \text{b) } \int_1^e \ln x dx & \quad \text{c) } \int_0^1 t e^{-t^2} dt & \quad \text{d) } \int_{-\pi}^{\pi} x \cos x dx & \quad \text{e) } \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx \\ \text{f) } \int_0^1 \sqrt{3x + 7} dx & \quad \text{g) } \int_{-3}^4 |x - 2| dx & \quad \text{h) } \int_0^1 3e^{5x+1} dx & \quad \text{i) } \int_0^{\pi} 2xt^3 \cos t^4 dt & \quad \text{j) } \int_1^2 \frac{1}{x} dx. \end{aligned}$$

2 Problems

2.1.

a) Find the area above the graph of $f(x) = x^2 - 4$ and below the x -axis.

b) Find the area between the graphs of $f(x) = x^2 + x + 1$ and $g(x) = 2x^2 + 5x + 4$, for $x \in [-3, 0]$.

2.2. Determine the domain, the intervals of monotony and the local extreme points of:

$$\text{a) } F(x) = \int_1^x \ln t dt \quad \text{b) } H(x) = \int_0^{x^2} e^{-t^2} dt.$$

2.3. Consider the function $f(x) = \int_{\pi}^{x^2} e^{-2t} dt$. Write Taylor's formula with order 1 of f around $x = 0$.

2.4. Consider the function $F(x) = \int_0^x tf(t)dt$, where f is continuous and strictly positive in \mathbb{R} . Prove that F has a local minimum at $x = 0$.

2.5. Consider the function $f(x) = \begin{cases} a - x & \text{if } x < 0 \\ \frac{1}{b+x} & \text{if } x \geq 0 \end{cases}$, with $b > 0$.

a) Determine the values of a and b for which f is continuous.

b) Take $a = b = 1$ and consider the function $F(x) = \int_{-1}^x f(t)dt$. Find $F(1)$. Moreover, show that F is invertible on $(0, +\infty)$.

2.6. Let $f(x) = \int_1^{x^2+1} \left(\frac{1+t}{t}\right) dt$.

a) Find $f(-1)$.

b) Determine the equation of the tangent line to the graph of f at $x = -1$.

2.7. Consider the function with domain D_f : $f(x) = \begin{cases} -x^2 - x + 1 & \text{if } x < 0 \\ e^{2x} & \text{if } x \geq 0 \end{cases}$.

a) For which values of $x \in D_f$ the function f is differentiable? Find $f'(x)$.

b) Determine $G(x) = \int_{-1}^x f(t)dt$, defined in $[-1, \infty)$.

2.8. Determine the function f , twice differentiable on \mathbb{R} , that satisfies: $f''(x) = 2 \cos x + xe^x$, $f'(0) = 2$, $f(0) = 1$

2.9. Without using anti-derivatives, compute:

a) $\lim_{x \rightarrow 0} \frac{\int_0^x \ln(t^2 + 1) dt}{x^3}$ b) $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \cos t dt}{x^2}$.

3 Additional exercises

3.1. Determine an anti-derivative of the following functions, in their respective domains:

- a) $x^2 e^x$ b) $x\sqrt{x+1}$ c) $x^3\sqrt{1+x^2}$ d) $2x \cos x$ e) $\sin^2 x$
f) $\ln(2x-1)$ g) $x^2 \ln x$ h) $\arctan x$ i) $\ln^2 x$ j) $e^x \cos x$.

3.2. Determine, by substitution, an anti-derivative of:

a) $\frac{x}{1+x^2}$ b) $\sqrt{1-\sin^2 x}$
c) $\frac{e^{\frac{x}{4}}}{1+e^{\frac{x}{10}}}$, with $x = 20 \ln t$ ($t > 0$) d) $\frac{\cos x}{\sin^6 x}$, with $x = \arcsin t$.

3.3. Study the convergence of the improper integrals and find their values whenever possible:

a) $\int_0^{+\infty} x e^{-x^2} dx$ b) $\int_0^{+\infty} \cos x dx$ c) $\int_0^{+\infty} \frac{\arctan x}{1+x^2} dx$
d) $\int_{-\infty}^0 \frac{1}{x^2+1} dx$ e) $\int_0^3 \frac{1}{x-3} dx$ f) $\int_0^2 \frac{2}{\sqrt{4-x^2}} dx.$

3.4. Find the area above the graph of $f(x) = \ln x$, for $x \in [0, 1]$, and below the line $y = 0$.

3.5. Book:

9.3: 4 to 6;

9.5: 2, 3;

9.6: 3;

9.7: 1, 4, 12.