Week 9: Chap. 6 – Elasticity, Implicit differentiation, Inverse function

1 Direct applications

1.1. Compute the elasticity in order to x of:
a) e^x b) $e^{\lambda x}$, with $\lambda \in \mathbb{R}$ c) $\dfrac{1}{x}$ d) $\cos(x^2)$.

1.2. Let $f(x) = \frac{1}{2}x^k h(x)$, with $k \in \mathbb{R}$ and h a differentiable function on its domain. Compute $El_x f(x)$.

1.3. Let f be twice differentiable on $\mathbb{R}$ such that: $2x^2 + 6xf(x) + [f(x)]^2 = 18$. Compute $\dfrac{df(x)}{dx}$ and $\dfrac{d^2f(x)}{dx^2}$.

1.4. For each of the following functions, discuss on which intervals they are invertible, find the inverse function and sketch the graph:
a) $\ln x$ b) x^2 c) $\dfrac{1}{x}$ d) $\sin x$ e) $\tan x$.

1.5. Compute, using the derivative of the inverse function theorem, the derivative at 1 (if it exists) of the inverse functions obtained in exercise 1.4.

1.6. Let $f(x) = x^2 e^x$.
a) Determine the intervals where f has an inverse.
b) Let $g(y)$ be the inverse function of $f(x)$ and x_0 a point where $f'(x_0) \neq 0$. Find the derivative of g at $y_0 = f(x_0)$.

2 Definitions and proofs

2.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ be differentiable on $\mathbb{R}$. Given a change Δx on x, the function feels a change $\Delta f(x) = f(x + \Delta x) - f(x)$.

Prove that $\displaystyle\lim_{\Delta x \to 0} \frac{\frac{\Delta f(x)}{f(x)}}{\frac{\Delta x}{x}} = \frac{x}{f(x)} f'(x)$.

2.2. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ be differenciable on their domain. By defining $u = g(x)$, show that $El_x(f \circ g)(x) = El_u f(u) \cdot El_x u$.

2.3. Let f be injective on $I \subseteq \mathbb{R}$, e $g = f^{-1}$. Write the equality relating f and g.

3 Problems and modelling

3.1. In a powder chocolate factory, the production cost f of chocolate, expressed in €/kg, depends on the price x of cacao, also in €/kg, as given by: $f(x) = x^2 + 3$, for $x \geq 0$. Consider a scenario where the price of cacao changed from 1 €/kg to 2 €/kg. Find the following:
a) The absolute change of the cacao price.
b) The absolute change of the production cost of chocolate.
c) The relative change of the cacao price.
d) The relative change of the production cost of chocolate.
e) The absolute rate of change of the production cost of chocolate against the increase of the cacao price.
f) The relative rate of change of the production cost of chocolate against the increase of the cacao price.
g) Consider now an infinitesimal increase dx of cacao x. Compute the absolute rate of change and the relative rate of change (elasticity) of the production cost of chocolate against the infinitesimal increase of cacao.

3.2. Imagine that the gasoline consumption c of a car depends on its speed v like: $c(v) = v^3 + 2v + 5$ (clearly, $v \geq 0$).
a) If the driver duplicates its speed, how does the gasoline consumption varies?
b) Let f the function that gives us the speed depending on the gasoline consumption: that is, $f[c(v)] = v$. Compute $f'(5)$.

3.3. Find the equation of the tangent line to the graph of f, defined implicitly by the equation $\sin[xf(x)] = f(x)$, at $(\frac{\pi}{2}, 1)$.

3.4. Let $g(x) = f[xg(x)]$ implicitly defined on $\mathbb{R}$. Knowing that $f'[g(1)] = 2$, find $g'(1)$?

3.5. Let $f\colon\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $f(x) = x^x$.
a) Is it exponencial?
b) Is it polynomial?
c) Use $e^{\ln x} = x$ to compute f'.

3.6. Book:
7.7: 2, 6.

4 Additional exercises

4.1. Find $L = \lim_{x\to 0^+} x^{5x}$:

a) L does not exist b) $L = 1$ c) $L = +\infty$ d) $L = 0$

4.2. Knowing that $f(x) = x^3 + 2x - 1$ admits an inverse function g and that $f(1) = 2$, find the slope of the tangent line to the graph of g at this point.

4.3. Let f be differentiable with $f(x) \neq 0$. Determine the elasticity of:

a) $x^5 f(x)$ b) $[f(x)]^{3/2}$ c) $x + \sqrt{f(x)}$ d) $\dfrac{1}{f(x)}$.

4.4. Differentiate:

a) $\tan^2(\arcsin x)$ b) $\arctan\left(x^2 - 1\right)$ c) $x^2 \arcsin x$ d) $\frac{1}{2}\arctan\left(e^{2x}\right)$.

4.5. Book:

7.7: 5, 9;
7.1: 1, 6, 7, 8, 10;
5.3: 3, 5, 7, 9, 11;
7.3: 1 - 3.