Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2008/2009

FINAL EXAM January 8, 2009

2 hours, 2 points per question

No consultation, no calculators

(1) Consider the function $y = f(x)$ implicitly defined by the equation

$$x + \ln y + y^2 = 3$$

and satisfying $f(2) = 1$. Determine the tangent line to the graph of f at the point $x = 2$.

(2) Compute the first order Taylor approximation of the function $f(x) = x^x$ around $a = 1$.

(3) Let f, g be differentiable functions in their domains. Assume also that $g(1) = e$, $g'(1) = 0$ and $f(1) = f'(1) \neq 0$. Find the elasticity of

$$F(x) = f\left(\frac{g(x)}{e^x}\right)$$

at the point $x = 1$.

Hint: Recall that the elasticity of a function h is given by the formula $El_x h = xh'(x)/h(x)$.

(4) For each a, compute the value of

$$\sum_{n=1}^{+\infty} \frac{a^n}{2^{n-1}}.$$

(5) Suppose that there is a function f satisfying the following properties:

$$f''(x) = \frac{e^x}{(e^x - 1)^2}, \qquad \lim_{x\to+\infty} f'(x) = -1, \qquad \lim_{x\to-\infty} f(x) = 0.$$

(a) Find its derivative f'.
(b) Find f.

(6) Let

$$f(x) = \frac{x^5}{x^6 + 5}.$$

(a) Find an antiderivative of f.
(b) Determine the area delimited by the graph of f and the lines $y = 0$, $x = 0$ and $x = 1$.

(7) Determine the values of a such that the following matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(8) Solve

$$\begin{cases} x + 2y - z = 1 \\ -x - y + 2z = 0 \\ x + y + 2z = 1 \end{cases}$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1st Semester 2008/2009

FINAL EXAM January 27, 2009

2 hours, 2 points per question

No consultation, no calculators

(1) Define a function F by

$$F(x) = \int_1^x \frac{\ln t}{t^3}\, dt.$$

(a) Compute $F(e)$.

(b) Determine the tangent line to the graph of F at $x = e$.

(2) Compute the second order Taylor approximation of the function $f(x) = x^x$ around $a = 1$.

(3) Let f, g be positive differentiable functions in their domains. Assume also that $g'(1) = 0$. Find the elasticity of

$$F(x) = f\left(\frac{1}{g(x)}\right) e^x$$

at the point $x = 1$.

Hint: Recall that the elasticity of a function h is given by the formula $El_x h = xh'(x)/h(x)$.

(4) For each a, compute the value of

$$\sum_{n=1}^{+\infty} (a-4)^n \frac{2}{3^{n-1}}.$$

(5) Let

$$f(x) = xe^{-4x}.$$

(a) Find an antiderivative of f.
(b) Compute the area delimited by $y = 0$, $x = 0$ and the graph of f.

(6) Without using antiderivatives, compute the limits:

$$\lim_{x\to 0} \frac{\int_0^x \ln(t^2+1)\,dt}{x^3} \quad \text{and} \quad \lim_{x\to 0} \frac{\int_0^{x^2} \cos(t)\,dt}{x^2}.$$

(7) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

(8) Solve

$$\begin{cases} x + y + z + w = 0 \\ x + 3y + 2z + 4w = 0 \\ 2x + y - w = 0 \end{cases}$$

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Departamento de Matemática

Mathematics 1

1st Semester 2009/2010

FINAL EXAM January 11, 2010

2 hours, No consultation, no calculators

(1) Compute:

(a) **(1.0)** the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}.$$

(b) **(1.0)** the value of

$$\lim_{x \to 0} \frac{3x^2}{2 - 2\cos x + 2x^2}.$$

(c) **(1.0)** for each $x \in \mathbb{R}$ the value of

$$\sum_{n=0}^{+\infty} \left(\frac{1-3x}{5} \right)^n.$$

(2) **(1.0)** Let $f(x) = \frac{3}{2} x^{-k} g(x)$ where g is a differentiable real-valued function in $\mathbb{R}$, and $k \in \mathbb{R}$. Show that the elasticity $El_x f$ of f at x is given by $-k + El_x g$.

(3) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 0 & 1 & \alpha \end{bmatrix}, \quad b = \begin{bmatrix} 1/3 \\ \beta \\ -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

(a) **(2.0)** Classify the system $A\,x = b$ in terms of the values of α and β.

(b) **(1.0)** Solve this system for $\alpha = -4$ and $\beta = -1$.

(c) **(0.5)** For which values of α the rows of A are linearly independent?

(4) Consider the function $f(x) = x \sin x$.

(a) **(1.0)** Write its Taylor polynomial of second order around the point 0.

(b) **(1.0)** The function f has a unique stationary point inside the interval $]-1, 1[$. Find it.

(c) **(1.0)** Classify the stationary point obtained in the previous question, by studying the second derivative.

(d) **(1.0)** Are there other extreme values of f in the interval $]-1, 1[$?

(e) **(1.5)** Compute one antiderivative of f.

(5) Given $k > 0$, consider the function

$$f(x) = \begin{cases} e^{x}, & x < 0 \\ e^{-kx}, & x \geq 0. \end{cases}$$

(a) **(1.0)** Find the domain of f and discuss the continuity of the function.

(b) **(1.0)** Using the definition of derivative, study the differentiability of f at the point 0.

(c) **(1.0)** Show that f is invertible in the open interval $]0, +\infty[$.

(d) **(1.0)** Take g to be the inverse function of f in $]0, +\infty[$. Find $g'(1/k)$.

(e) **(1.5)** Compute the area between the graph of f and the x-axis.

(6) **(1.5)** Let $f, g\colon \mathbb{R} \to \mathbb{R}$ differentiable functions and $a \in \mathbb{R}$. Prove that

$$\frac{d}{dx} \int_{a}^{g(x)} f(t)\, dt = f(g(x)) \, \frac{dg}{dx}(x).$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2009/2010

FINAL EXAM January 27, 2010

2 hours, no consultation, no calculators

(1) **(1.0)** Let $u = (1, 0, 2, 0)$ and $v = (\alpha, 1, 1, \pi)$ with $\alpha \in \mathbb{R}$. Determine the value of α for which u and v are orthogonal.

(2) Compute:

(a) **(1.0)**

$$\frac{d}{dx} \int_0^{x^2} e^t \, dt.$$

(b) **(1.0)**

$$\sum_{n=0}^{+\infty} \left[\left(-\frac{1}{2} \right)^n + \left(\frac{1}{2} \right)^n \right].$$

(3) **(1.0)** Let A, B, P and X be matrices $n \times n$ such that $\det P \neq 0$. Find the solution of the equation $PX + AB = 0$ with respect to X.

(4) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\alpha \\ -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ \beta \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

(a) **(2.0)** Classify the system $A\,x = b$ in terms of the values of α and β.

(b) **(1.0)** Solve this system for $\alpha = 0$ and $\beta = 1$.

(5) Consider the vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, k)$ and $v_3 = (0, 0, 1)$, where $k \in \mathbb{R}$.

(a) **(1.0)** Determine if these vectors are linearly independent.

(b) **(0.5)** Compute the distance between v_1 and v_2.

(6) Consider the function $f(x) = x^5$.

(a) **(1.5)** Sketch the graph of f, determining and classifying its stationary points.

(b) **(1.0)** Given $k > 0$, compute the area between the graph of f and the x-axis for $x \in [-k, k]$.

(c) **(1.0)** Given $k > 0$, compute $\int_{-k}^{k} f(x)\,dx$.

(d) **(2.0)** Let $g\colon \mathbb{R} \to \mathbb{R}$ two-times differentiable on $\mathbb{R}$ and such that $g(0) \neq 0$. Knowing that 0 is a stationary point of g and that $g''(0) > 0$, show that 0 is a local minimum of $h = f \circ g$.

(7) Let $f(x) = \sqrt{x}$.

(a) **(1.5)** Find the linear approximation to f around 1 and use it to obtain an approximation of $\sqrt{1.1}$.

(b) **(1.5)** Compute $\int_0^{\pi^2} \frac{\cos f(x)}{f(x)}\,dx$.

(c) **(1.5)** Determine

$$\lim_{x \to 0^+} \frac{\sin f(x)}{f(x)}.$$

(8) **(1.5)** Let $f\colon \mathbb{R} \to \mathbb{R}^+$ a differentiable function on its domain and $p \in \mathbb{R}$. Show that

$$El_x[f(x)]^p = p\,El_x f(x).$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM

January 4, 2011

2 hours, No consultation, No calculators

1. (1.5) Find k that minimizes the distance between $\vec{u} = (-1, 1, k, 0)$ and $\vec{v} = (2, 0, 0, -7)$.

2. (1.5) For which values of $x \in \mathbb{R}$ the series $\sum_{n=0}^{+\infty} 7(\cos x)^n$ converges?

3. (1.5) Find $\lim_{x \to 0^+} x^{5x}$.

4. (1.5) Compute the following determinant:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \end{vmatrix},$$

   where $\alpha, \beta, \gamma \in \mathbb{R}$.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:

$$\begin{cases} x + 2y + z = 0 \\ -x - y + \alpha z = 3 \\ -2x - 3y + z = \beta \end{cases}$$

   a. (2.5) Classify this system depending on α and β.
   b. (0.5) Solve this system for $\alpha = 2$ and $\beta = 3$.

6. (0.5) Define rank of a matrix.

7. (0.5) Consider the vectors $\vec{a} = (0,2,0)$, $\vec{b} = (2,x,0)$ and $\vec{c} = (0,0,y)$, where $x, y \in \mathbb{R}$. Find for which values of x and y the vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

8. (1.5) Let A be an invertible matrix. Show that $(A^{-1})^{-1} = A$.

9. Consider the function $f(x) = xe^x + 3$.
   a. (0.5) Find the domain of f and discuss its continuity.
   b. (1.0) Determine the stationary points of f.
   c. (1.0) Determine the extreme points of f using the second derivative.
   d. (1.0) Discuss if the above extreme points are global.
   e. (1.0) Write the quadratic approximation of f around $x = 0$.
   f. (0.5) Find the intervals where f is invertible.
   g. (1.0) Determine the derivative of the inverse function f^{-1} at 3.

10. (1.5) Consider $f(x) = -e^{-x}$ and $g(x) = e^{-x}$. Compute the area between the graphs of f and g, with $x > 0$.

11. (1.0) Let $f\colon \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain, and $p \in \mathbb{R}$. Prove that $\mathrm{El}_x\left[f(x)^p\right] = p\ \mathrm{El}_x\left[f(x)\right]$.

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM

January 24, 2011

2 hours, No consultation, No calculators

1. (1.5) Let $\vec{u} = (1, 2, 0, k)$ and $\vec{v} = (3, -1, -1, -1)$. Find k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

2. (1.5) Let $f \colon \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain. Compute $El_x \dfrac{1}{f(x)}$.

3. (1.5) For each value of $\beta \in \mathbb{R}$ find

$$\lim_{x \to 1} \frac{x^3 - 3\beta x + 3\beta - 1}{(x-1)^2}.$$

4. (1.5) Decide if the function $f(x) = \sqrt{x + \alpha}$, with $\alpha \in \mathbb{R}$, is linear or non-linear.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:

$$\begin{cases} x + y + z = 1 \\ 2x + 5z = 1 \\ x - y + \alpha z = \beta \end{cases}$$

   a. (2.0) Classify this system depending on α and β.
   b. (1.0) Solve this system for $\alpha = 0$ and $\beta = 0$ using Cramer's rule.

6. (1.0) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

7. (1.0) Let A, B, C, X be $n \times n$ matrices and let I be the $n \times n$ identity matrix. Assuming that C and $A - B$ are invertible, solve the equation $AXC = BXC + I$ with respect to X.

8. (1.5) Consider the series

$$\sum_{n=0}^{+\infty} \left(\frac{3x + 2}{3} \right)^n .$$

Discuss for which values of x the series converges, and compute its sum whenever possible.

9. Consider the function $g(x) = e^{(x^2+1)}$.
   a. (0.5) Find the domain of g and discuss its continuity.
   b. (1.0) Determine the stationary points of g.
   c. (1.0) Determine the extreme points of g using the second derivative.
   d. (0.5) Study the concavity of g.
   e. (0.5) Discuss if the above extreme points are global.
   f. (1.5) Compute $\displaystyle\int_0^1 xg(x)\,dx$.

10. Recall that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

   a. (1.5) Estimate the value of $\arctan(0.1)$.
   b. (1.0) Give an upper bound for the approximation error.