Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1 2011/2012

FINAL EXAM January 3, 2012

2 hours, no consultation, no calculators

1. (1.5) Let A, B and C be $n \times n$ invertible matrices. Find the solution X of the equation $A^{-1}XB + B = C$.

2. (1.5) Consider a differentiable function f on $\mathbb{R}$ such that $f(1) = 1$ and $f'(1) = 3$. Determine the elasticity of $h(x) = f(e^{\cos x})$ with respect to x at $\pi/2$.

3. (1.5) Compute the area between the functions $\sqrt{x}$ and $-\sqrt{x}$ when $x \in [0, 1]$.

4. (1.5) Let $\vec{u} = (1, 0, 2, \pi)$ and $\vec{v} = (2, 0, k, 1)$ with $k \in \mathbb{R}$. For which value of k are these vectors orthogonal?

5. When the market is in equilibrium, the production x of wheat and its demand y, and the production z of bread and its demand t satisfy the following equations:

$$\begin{cases} x + 4y + 4t &= 350 \\ x - 4y &= -50 \\ y + z + 4t &= 250 \\ z - \frac{3}{2}t &= -50, \end{cases}$$

   where all the amounts are expressed in tons.

   a. (1.5) Write this system using matrices and classify it by studying the rank.

   b. (1.0) Determine the production of wheat and bread that satisfy this equilibrium.

6. (1.5) Find for which values of $k \in \mathbb{R}$ the vectors $\vec{u} = (1, k, 1, k)$, $\vec{v} = (k, 1, k, 1)$ and $\vec{w} = (0, 1, 1, 1)$ and $\vec{t} = (0, 0, 0, 2)$ are linearly independent.

7. (1.5) Let A be an invertible $n \times n$ matrix and B be a matrix constructed by multiplying a row (or a column) of A by $\lambda \in \mathbb{R}$. Show that $|B| = \lambda\,|A|$.

8. (2.0) An investor has two investment projects:
   - the first one consists on a bond with yearly payments that correspond after ten years to the value
   $$\sum_{t=1}^{10} \frac{1200}{2^t}.$$
   - the second one generates a profit in continuous time, and after 10 years it yields
   $$\int_0^{10} 120\,t\,e^{t/2}\,dt.$$

   Which of the two investment projects is more profitable after 10 years?

9. A traditional ice cream shop produces each day up to 80 Kg of ice cream. According to the specific characteristics of this product, the profit function of the shop is given by:
   $$L(x) = \begin{cases} x^3 + 3x - 10, & 0 \leq x \leq 3 \\ -x^2 + 10x + 5, & 3 < x \leq 8, \end{cases}$$
   where x represents the tens of Kg of ice cream.
   a. (1.0) Discuss the continuity of L.
   b. (1.0) Study the differentiability of L at $x = 3$ using the definition. Write the function L'.
   c. (1.5) Determine the stationary points of L and classify them using the second derivative.
   d. (1.5) Compute the amount of ice cream (in tens of Kg) that maximizes and that minimizes the profit of the shop.

10. (1.5) Let
   $$f(x) = \int_0^x e^{t^2}\,dt.$$
   Write the quadratic approximation of f around $x = 1$. *Note:* It is not necessary to find $f(1)$ explicitly.