Mathematics I

Exercises with solutions

1 Linear Algebra

Vectors and Matrices

1.1. Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

Determine the following matrices:

a) $A + B$; b) $A - B$; c) AB; d) BA; e) $(AB)C$; f) $A(BC)$

Solution:

a) $\begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 3 & -5 \\ 1 & -2 & -3 \\ -1 & -1 & -2 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 1 & -3 \end{bmatrix}$ e) $\begin{bmatrix} 23 & 8 & 25 \\ 92 & -28 & 76 \\ 4 & -8 & -4 \end{bmatrix}$

f) idem

1.2. Let

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}.$$

Compute A^T, B^T, $(A+B)^T$, AB, $(AB)^T$, $B^T A^T$, $A^T B^T$ and identify the properties of matrix transposition that are illustrated by the calculations.

Solution:

$A^T = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$; $B^T = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$; $(A+B)^T = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$; $AB = \begin{bmatrix} 4 & 10 \\ 10 & 8 \end{bmatrix}$; $(AB)^T = \begin{bmatrix} 4 & 10 \\ 10 & 8 \end{bmatrix}$;

$A^T B^T = \begin{bmatrix} -2 & 4 \\ 10 & 14 \end{bmatrix}$

1.3. Determine a so that the following matrix is symmetric

$$\begin{bmatrix} a & a^2 - 1 & -3 \\ a + 1 & 2 & a^2 + 4 \\ -3 & 4a & -1 \end{bmatrix}$$

Solution: $a = 2$.

Inverse Matrix

1.4. Show that the inverse of $\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{bmatrix}$ and that the inverse of $\begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.5. Determine a and b so that the matrix A is the inverse of the matrix B, where $A = \begin{bmatrix} 2 & -1 & -1 \\ a & \frac{1}{4} & b \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$

Solution:

$$\begin{cases} a + b = 0 \\ 2a + \frac{1}{4} + 3b = 1 \end{cases} \Leftrightarrow \begin{cases} a = -\frac{3}{4} \\ b = \frac{3}{4} \end{cases}$$

1.6. Using elementary operations, determine the inverse of the following matrices:

a) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & -2 & 1 \\ 4 & 16 & 8 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ g) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 5 & 8 \\ -9 & -6 & 3 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

Solution: a) $\begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$ b) $\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$ c) $\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{7}{8} & -\frac{1}{4} & \frac{1}{32} \\ \frac{5}{4} & \frac{1}{2} & \frac{1}{16} \end{bmatrix}$

e) $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ f) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ g) $\begin{bmatrix} \frac{21}{34} & -\frac{2}{17} & \frac{11}{102} \\ -\frac{23}{34} & \frac{3}{17} & -\frac{25}{102} \\ \frac{1}{2} & 0 & \frac{1}{6} \end{bmatrix}$ h) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Linear dependence and linear independence. Rank of a matrix

1.7. In the corresponding vector spaces, determine if the following vectors are linearly independent,

a) $(3, 1)$ and $(4, 2)$ in $\mathbb{R}^2$.

b) $(3, 1)$, $(4, -2)$ and $(7, 2)$ in $\mathbb{R}^2$.

c) $(0, -3, 1)$, $(2, 4, 1)$ and $(-2, 8, 5)$ in $\mathbb{R}^3$.

d) $(-1, 2, 0, 2)$, $(5, 0, 1, 1)$ and $(8, -6, 1, -5)$ in $\mathbb{R}^4$.

Solution: a) Linearly independent; b) Linearly dependent: $\begin{cases} \alpha_1 = 22\alpha_2 \\ \alpha_3 = -10\alpha_2 \end{cases}$, $\alpha_2 \in \mathbb{R}$;
c) Linearly independent; d) Linearly dependent: $\begin{cases} \alpha_1 = 3\alpha_3 \\ \alpha_2 = -\alpha_3 \end{cases}$, $\alpha_3 \in \mathbb{R}$.

1.8. Discuss the linear independence of the vectors: $a = (1, -2)$ e $b = (\alpha, -1)$ in $\mathbb{R}^2$, depending on the value of the real parameter α

Solution: If $\alpha \neq \frac{1}{2} \Longrightarrow$ Linearly independent; if $\alpha = \frac{1}{2} \Longrightarrow$ Linearly dependent: $x_1 = -\frac{x_2}{2}$, $x_2 \in \mathbb{R}$.

1.9. Determine the rank of the following matrices

a) $\begin{bmatrix} 1 & 2 \\ 8 & 16 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{bmatrix}$

e) $\begin{bmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{bmatrix}$ f) $\begin{bmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{bmatrix}$

Solution: a) r(M)=1; b) r(M)=2; c) r(M)=2; d) r(M)=3; e) r(M)=2; f) r(M)=3.

1.10. Depending on the values of the parameters, determine the rank of the following matrices

a) $\begin{bmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{bmatrix}$ b) $\begin{bmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{bmatrix}$ c) $\begin{bmatrix} 1 & x & y & 0 \\ 0 & z & w & 1 \\ 1 & x & y & 1 \\ 0 & z & w & 1 \end{bmatrix}$

Solution: a) If $x = -1$ or $x = 2$, then $r(M) = 2$. Otherwise, $r(M) = 3$;
b) If $t = \pm 2$ or $t = -4$, then $r(M) = 2$. Otherwise, $r(M) = 3$;
c) If $z = 0$ and $w = 0$, then $r(M) = 2$. Otherwise, $r(M) = 3$.

1.11. Find an example with 2×2 matrices, that illustrate the fact that in general we have $r(AB) \neq r(BA)$.

Solution: A=$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, B=$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; AB=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, BA=$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $r(AB) = 0 \neq r(BA) = 1$.

Determinants

1.12. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Compute $|A|$, $|A^T|$, $|AB|$, $|B|$, $|B^T|$, $|A^T B^T|$ e and identify the properties that are illustrated by the calculations.

Solution: $|A| = -2 = |A^T|$; $|B| = |B^T| = -2$; $|AB| = 4 = |A^T B^T|$;

1.13. Compute the following determinants:

$$\text{a)} \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} \quad \text{b)} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{vmatrix} \quad \text{c)} \begin{vmatrix} 2 & 1 & 3 & 3 \\ 3 & 2 & 1 & 6 \\ 1 & 3 & 0 & 9 \\ 2 & 4 & 1 & 12 \end{vmatrix} \quad \text{d)} \begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix}$$

$$\text{e)} \begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{vmatrix} \quad \text{f)} \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 4 & 0 & 3 & 4 \\ 6 & 2 & 3 & 1 & 2 \end{vmatrix} \quad \text{g)} \begin{vmatrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 3 & 2 & 5 & -3 \end{vmatrix}$$

Solution: a) 2 b) 30 c) 0 d) $-abc$ e) $abcd$ f) 360 g) 4.

1.14. Show that the following determinants are equal to zero.

$$\text{a)} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 8 \end{vmatrix} \quad \text{b)} \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \quad \text{c)} \begin{vmatrix} x-y & x-y & x^2-y^2 \\ 1 & 1 & x+y \\ y & 1 & x \end{vmatrix}$$

Solution: a) $C_2 = 2C_1$ b) $(a+b+c)C_1 = C_2 + C_3$ c) $(x-y)L_2 = L_1$

1.15. Consider the following matrices $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $C = A^{\mathsf{T}} \times B$. Compute the determinant of the matrix C.

Solution: $|C| = 0$

1.16. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute the determinant of the matrix $C = (B \times A)^T$.

Solution: $|C| = 0$

1.17. Determine the values of a such that the matrix $A = \begin{bmatrix} a & 1 & 2 \\ 0 & 1-a & a \\ 0 & 3 & 0 \end{bmatrix}$ has determinant equal to -12.

Solution: $a = \pm 2$

Systems of equations

1.18. Using Cramer's Rule, solve the following systems

a) $\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3 \end{cases}$ b) $\begin{cases} x + y = 3 \\ x + z = 2 \\ y + z + u = 6 \\ y + u = 1 \end{cases}$

c) Find p so that the following system has a solution. Find the solutions.

$$\begin{cases} px + y = 1 \\ x - y + z = 0 \\ 2y - z = 3 \end{cases}$$

Solution: a) $x = 1, y = -2, z = 2$ b) $x = -3, y = 6, z = 5, u = -5$
c) The system has a solution if and only if $p \neq 1$. The solution is $x = -\frac{2}{p-1}, y = \frac{3p-1}{p-1}, z = \frac{3p+1}{p-1}$.

1.19. Show that the following system has a unique solution for all b_1, b_2, b_3 and find it.

$$\begin{cases} 3x_1 + x_2 = b_1 \\ x_1 - x_2 + 2x_3 = b_2 \\ 2x_1 + 3x_2 - x_3 = b_3 \end{cases}$$

Solution: The coefficient matrix of the system is invertible, which means that the system has a unique solution for any b_1, b_2, b_3. The solution is: $x_1 = \frac{1}{2}b_1 - \frac{1}{10}b_2 - \frac{1}{5}b_3, x_2 = -\frac{1}{2}b_1 + \frac{3}{10}b_2 + \frac{3}{5}b_3, x_3 = -\frac{1}{2}b_1 + \frac{7}{10}b_2 + \frac{2}{5}b_3$.

1.20. Show that the homogeneous system

$$\begin{cases} ax + by + cz = 0 \\ bx + cy + az = 0 \\ cx + ay + bz = 0 \end{cases}$$

has non trivial solutions if and only if $a^3 + b^3 + c^3 - 3abc = 0$.

Solution: A homogeneous system has no non-trivial solutions only if the coefficient matrix of the system is singular, which occurs if its determinant is equal to zero. In this case the determinant of the coefficient matrix is $a^3 + b^3 + c^3 - 3abc$, from which follows the intended result.

1.21. Discuss the solutions of the system

$$\begin{cases} x + 2y + 3z = 1 \\ -x + ay - 21z = 2 \\ 3x + 7y + az = b \end{cases}$$

depending on the parameters a and b.

Solution: If $a \neq 0 \wedge a \neq 7$ the system has a unique solution regardless of the parameter b. If $a = 0$, the system has a solution only if $b = \frac{9}{2}$. If $a = 7$, the system has a solution only if $b = \frac{10}{3}$.

1.22. Consider the system

$$\begin{cases} x_1 + x_2 + x_3 = 2q \\ 2x_1 - 3x_2 + 2x_3 = 4q \\ 3x_1 - 2x_2 + px_3 = q \end{cases}$$

where p e q are arbitrary constants. Determine for which values of the constants the system has: a) a unique solution; b) Infinite solutions; c) No solutions.

Solution: a) $p \neq 3$ b) $p = 3 \wedge q = 0$ c) $p = 3 \wedge q \neq 0$

1.23. Show that the following system

$$\begin{cases} 2x + 3y = k \\ x + cy = 1 \end{cases}$$

has a unique solution, except for a particular value c^* of c. Determine that solution. Also show that for $c = c^*$, the system has no solutions except for a particular value k^* of k. Find the solution when $k = k^*$.

Solution: The determinant of the coefficient matrix of the system is equal to zero only when $c = \frac{3}{2}$. So, when $c \neq \frac{3}{2}$, the system has a unique solution which is $y = -\frac{k-2}{-3+2c}$, $x = \frac{-3+kc}{-3+2c}$. When $c = \frac{3}{2}$ the system has a solution only if $k = 2$, and in that case the solution is $\left(x, \frac{2}{3}(1-x)\right)$.

1.24. Find the values of a e b so that the system $\begin{cases} 3x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + az = b - 3 \end{cases}$ is possible and has infinitely many solutions.

Solution: The system is possible and has infinitely many solutions (1 degree of freedom) if $a = \frac{7}{5} \wedge b = 3$

1.25. Classify the system of equations, depending on the parameter α

$$\begin{cases} x + 2y + z + w = 0 \\ 2x + 4y + 2z + 3w = 1 \\ x + 2y + z + 2w = \alpha \end{cases}$$

Solution: If $\alpha = 1$,The system is possible and has infinitely many solutions (1 degree of freedom).
If $\alpha \neq 1$, the system has no solutions.

1.26. Discuss the existence of solutions for the following systems, finding, whenever possible, the number of degrees of freedom and the solutions

a) $\begin{cases} -2x - 3y + z = 3 \\ 4x + 6y - 2z = 1 \end{cases}$ b) $\begin{cases} x + y - z + w = 2 \\ 2x - y + z - 3w = 1 \end{cases}$ c) $\begin{cases} x - y + 2z + w = 1 \\ 2x + y - z + 3w = 3 \\ x + 5y - 8z + w = 1 \\ 4x + 5y - 7z + 7w = 7 \end{cases}$

d) $\begin{cases} x + y + 2z + w = 5 \\ 2x + 3y - z - 2w = 2 \\ 4x + 5y + 3z = 7 \end{cases}$ e) $\begin{cases} x - y + z = 0 \\ x + 2y - z = 0 \\ 2x + y + 3z = 0 \end{cases}$ f) $\begin{cases} x + y + z + w = 0 \\ x + 3y + 2z + 4w = 0 \\ 2x + y - w = 0 \end{cases}$

g) $\begin{cases} x + 2y - z = 1 \\ -x - y + 2z = 0 \\ x + y + 2z = 1 \end{cases}$ h) $\begin{cases} x + y - z + w = 1 \\ x - y + z - w = 0 \\ x + 2y - z = 0 \\ x - y - 2z = 1 \end{cases}$ i) $\begin{cases} 2x + y - z + w = 1 \\ 3x - 2y + 2z - 3w = 2 \\ 5x + y - z + 2w = -1 \\ 2x - y + z - 3w = 4 \end{cases}$

j) $\begin{cases} x + y + z = 1 \\ x - y + z = 0 \\ y + 2z = 1 \\ 2x + y + 4z = 2 \end{cases}$ k) $\begin{cases} x + y + 2z + w = 1 \\ y + 3z + 3w = 2 \\ -x + z + 2w = 1 \\ 2x + y + z - w = 0 \end{cases}$ l) $\begin{cases} 2x + y + 3z - w = 1 \\ 4x - y + 7z - 7w = -5 \\ x + 2y + z + 2w = 3 \end{cases}$

m) $\begin{cases} x - y + z = 0 \\ y - z + w = 2 \\ x + y + z + w = -1 \\ y + w = 3 \end{cases}$ n) $\begin{cases} x + 2y - z + w = 1 \\ 2x + y + z - w = -2 \\ x + z + w = 1 \\ 2x + 5y - 3z + 5w = 5 \end{cases}$ o) $\begin{cases} y - z + w = 4 \\ -x + y + 2z + w = 1 \\ -x + z + 2w = -2 \\ -2x + y + 7z - w = -3 \end{cases}$

p) $\begin{cases} x - y + z + 2w = 1 \\ -x + 2y + z - w = 1 \\ y + 2z + w = 2 \\ 2x - 3y + 3w = 0 \end{cases}$ q) $\begin{cases} x + y - z + w = 2 \\ 2x + y - z + w = 1 \\ x + 2y - 2z + 2w = 5 \\ x - y + z = 1 \end{cases}$ r) $\begin{cases} x - y + 2z + w = 1 \\ 3x + 2z + 2w = 2 \\ 2x + y + w = 1 \\ x + 8y - 10z - 2w = -2 \end{cases}$

Solution:

a) The system has no solutions

b)The system has infinitely many solutions. (2 degrees of freedom).
Solutions of the form: $\left(1+\frac{2}{3}w, 1-\frac{5}{3}w+z, z, w\right)$.

c) The system has infinitely many solutions. (1 degree of freedom). Solutions of the form: $\left(-\frac{1}{3}z, \frac{5}{3}z, z, 1\right)$.

d) The system has no solutions

e) The system has a unique solution. Solution: $(0, 0, 0)$.

f) The system has infinitely many solutions. (1 degree of freedom). Solutions of the form: $(w, -w, -w, w)$.

g)The system has a unique solution. Solution: $\left(-\frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right)$.

h)The system has a unique solution. Solution: $\left(\frac{1}{2}, -\frac{3}{10}, -\frac{1}{10}, \frac{7}{10}\right)$.

i) The system has no solutions.

j) The system has a unique solution. Solution: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

k) The system has infinite many solutions. (2 degrees of freedom).
Solutions of the form: $(-1+z+2w, 2-3z-3w, z, w)$.

l) The system has no solutions.

m) The system has a unique solution. Solution: $(-5, -4, 1, 7)$.

n) The system has no solutions.

o) The system has infinitely many solutions. (1 degree of freedom).
Solutions of the form: $\left(\frac{3}{2}+3w, \frac{7}{2}, -\frac{1}{2}+w, w\right)$.

p) The system has infinitely many solutions. (2 degrees of freedom).
Solutions of the form: $(3-3z-3w, 2-2z-w, z, w)$.

q) The system has infinitely many solutions. (1 degree of freedom). Solutions of the form: $(-1, -2+z, z, 5)$.

r) The system has infinitely many solutions. (2 degrees of freedom).
Solutions of the form :$\left(\frac{2}{3}-\frac{2}{3}w-\frac{2}{3}z, -\frac{1}{3}+\frac{1}{3}w+\frac{4}{3}z, z, w\right)$.

1.27. Discuss the following linear systems, depending on the parameters a,b e c:

a) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ x-y-z=a \end{cases}$ b) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-2y+az=2 \end{cases}$ c) $\begin{cases} y+az=0 \\ x+by=0 \\ by+az=1 \end{cases}$

d) $\begin{cases} x+y+z=1 \\ x-y+2z=a \\ 2x+bz=2 \end{cases}$ e) $\begin{cases} 2x+y=b \\ 3x+2y+z=0 \\ x+ay+z=2 \end{cases}$ f) $\begin{cases} ax+y-z+aw=0 \\ (a+1)y+z+w=1 \\ -x+y+(a+1)w=b \end{cases}$

g) $\begin{cases} ax+y+(a+1)z=b \\ x+ay+z=1 \\ ax+y-z=0 \end{cases}$ h) $\begin{cases} x+y+z=2 \\ ax+z=2 \\ 3x+y+3z=6 \end{cases}$ i) $\begin{cases} x+y=a \\ x+2y=a^2 \\ x+3y=a^3 \end{cases}$

j) $\begin{cases} -2x+(a+3)y-bz=-3 \\ x+bz=1 \\ 2x+4y+3bz=-b \end{cases}$ m) $\begin{cases} x-y+2z=1 \\ x+az=1 \\ x+y+2z=b \\ 2x-y+(a+2)z=2 \end{cases}$ n) $\begin{cases} 2x+4y+bz=2 \\ x+(a+2)y=1 \\ x+by+az=1 \\ x+2y=c \end{cases}$

Solution:

a) For all a: the system has a unique solution.

b) If $a \neq 2$ the system has a unique solution; If $a = 2$, the system has infinitely many solutions (with 1 degree of freedom).