Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2008/2009

FINAL EXAM January 8, 2009

2 hours, 2 points per question

No consultation, no calculators

(1) Consider the function $y = f(x)$ implicitly defined by the equation

$$x + \ln y + y^2 = 3$$

and satisfying $f(2) = 1$. Determine the tangent line to the graph of f at the point $x = 2$.

(2) Compute the first order Taylor approximation of the function $f(x) = x^x$ around $a = 1$.

(3) Let f, g be differentiable functions in their domains. Assume also that $g(1) = e$, $g'(1) = 0$ and $f(1) = f'(1) \neq 0$. Find the elasticity of

$$F(x) = f\left(\frac{g(x)}{e^x}\right)$$

at the point $x = 1$.

Hint: Recall that the elasticity of a function h is given by the formula $El_x h = xh'(x)/h(x)$.

(4) For each a, compute the value of

$$\sum_{n=1}^{+\infty} \frac{a^n}{2^{n-1}}.$$

(5) Suppose that there is a function f satisfying the following properties:

$$f''(x) = \frac{e^x}{(e^x - 1)^2}, \quad \lim_{x \to +\infty} f'(x) = -1, \quad \lim_{x \to -\infty} f(x) = 0.$$

(a) Find its derivative f'.
(b) Find f.

(6) Let

$$f(x) = \frac{x^5}{x^6 + 5}.$$

(a) Find an antiderivative of f.
(b) Determine the area delimited by the graph of f and the lines $y = 0$, $x = 0$ and $x = 1$.

(7) Determine the values of a such that the following matrix is invertible:

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(8) Solve

$$\begin{cases} x + 2y - z = 1 \\ -x - y + 2z = 0 \\ x + y + 2z = 1 \end{cases}$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1st Semester 2008/2009

FINAL EXAM January 27, 2009

2 hours, 2 points per question

No consultation, no calculators

(1) Define a function F by

$$F(x) = \int_1^x \frac{\ln t}{t^3}\, dt.$$

(a) Compute $F(e)$.

(b) Determine the tangent line to the graph of F at $x = e$.

(2) Compute the second order Taylor approximation of the function $f(x) = x^x$ around $a = 1$.

(3) Let f, g be positive differentiable functions in their domains. Assume also that $g'(1) = 0$. Find the elasticity of

$$F(x) = f\left(\frac{1}{g(x)}\right) e^x$$

at the point $x = 1$.

Hint: Recall that the elasticity of a function h is given by the formula $El_x h = xh'(x)/h(x)$.

(4) For each a, compute the value of

$$\sum_{n=1}^{+\infty} (a-4)^n \frac{2}{3^{n-1}}.$$

(5) Let

$$f(x) = xe^{-4x}.$$

(a) Find an antiderivative of f.

(b) Compute the area delimited by $y = 0$, $x = 0$ and the graph of f.

(6) Without using antiderivatives, compute the limits:

$$\lim_{x \to 0} \frac{\int_0^x \ln(t^2+1)\,dt}{x^3} \quad \text{and} \quad \lim_{x \to 0} \frac{\int_0^{x^2} \cos(t)\,dt}{x^2}.$$

(7) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

(8) Solve

$$\begin{cases} x + y + z + w = 0 \\ x + 3y + 2z + 4w = 0 \\ 2x + y - w = 0 \end{cases}$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2009/2010

FINAL EXAM January 11, 2010

2 hours, No consultation, no calculators

(1) Compute:

(a) **(1.0)** the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}.$$

(b) **(1.0)** the value of

$$\lim_{x\to 0} \frac{3x^2}{2-2\cos x+2x^2}.$$

(c) **(1.0)** for each $x \in \mathbb{R}$ the value of

$$\sum_{n=0}^{+\infty} \left(\frac{1-3x}{5}\right)^n.$$

(2) **(1.0)** Let $f(x) = \frac{3}{2}x^{-k}g(x)$ where g is a differentiable real-valued function in $\mathbb{R}$, and $k \in \mathbb{R}$. Show that the elasticity $El_x f$ of f at x is given by $-k + El_x g$.

(3) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 0 & 1 & \alpha \end{bmatrix}, \quad b = \begin{bmatrix} 1/3 \\ \beta \\ -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

(a) **(2.0)** Classify the system $A\,x = b$ in terms of the values of α and β.

(b) **(1.0)** Solve this system for $\alpha = -4$ and $\beta = -1$.

(c) **(0.5)** For which values of α the rows of A are linearly independent?

(4) Consider the function $f(x) = x \sin x$.

(a) **(1.0)** Write its Taylor polynomial of second order around the point 0.

(b) **(1.0)** The function f has a unique stationary point inside the interval $]-1, 1[$. Find it.

(c) **(1.0)** Classify the stationary point obtained in the previous question, by studying the second derivative.

(d) **(1.0)** Are there other extreme values of f in the interval $]-1, 1[$?

(e) **(1.5)** Compute one antiderivative of f.

(5) Given $k > 0$, consider the function

$$f(x) = \begin{cases} e^x, & x < 0 \\ e^{-kx}, & x \geq 0. \end{cases}$$

(a) **(1.0)** Find the domain of f and discuss the continuity of the function.

(b) **(1.0)** Using the definition of derivative, study the differentiability of f at the point 0.

(c) **(1.0)** Show that f is invertible in the open interval $]0, +\infty[$.

(d) **(1.0)** Take g to be the inverse function of f in $]0, +\infty[$. Find $g'(1/k)$.

(e) **(1.5)** Compute the area between the graph of f and the x-axis.

(6) **(1.5)** Let $f, g \colon \mathbb{R} \to \mathbb{R}$ differentiable functions and $a \in \mathbb{R}$. Prove that

$$\frac{d}{dx} \int_a^{g(x)} f(t)\, dt = f(g(x)) \frac{dg}{dx}(x).$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2009/2010

FINAL EXAM January 27, 2010

2 hours, no consultation, no calculators

(1) **(1.0)** Let $u = (1,0,2,0)$ and $v = (\alpha,1,1,\pi)$ with $\alpha \in \mathbb{R}$. Determine the value of α for which u and v are orthogonal.

(2) Compute:

(a) **(1.0)**

$$\frac{d}{dx}\int_0^{x^2} e^t\,dt.$$

(b) **(1.0)**

$$\sum_{n=0}^{+\infty}\left[\left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right].$$

(3) **(1.0)** Let A, B, P and X be matrices $n \times n$ such that $\det P \neq 0$. Find the solution of the equation $PX + AB = 0$ with respect to X.

(4) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\alpha \\ -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ \beta \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

(a) **(2.0)** Classify the system $A\,x = b$ in terms of the values of α and β.

(b) **(1.0)** Solve this system for $\alpha = 0$ and $\beta = 1$.

(5) Consider the vectors $v_1 = (1,0,0)$, $v_2 = (0,1,k)$ and $v_3 = (0,0,1)$, where $k \in \mathbb{R}$.
  (a) **(1.0)** Determine if these vectors are linearly independent.
  (b) **(0.5)** Compute the distance between v_1 and v_2.

(6) Consider the function $f(x) = x^5$.
  (a) **(1.5)** Sketch the graph of f, determining and classifying its stationary points.
  (b) **(1.0)** Given $k > 0$, compute the area between the graph of f and the x-axis for $x \in [-k, k]$.
  (c) **(1.0)** Given $k > 0$, compute $\int_{-k}^{k} f(x)\,dx$.
  (d) **(2.0)** Let $g\colon \mathbb{R} \to \mathbb{R}$ two-times differentiable on $\mathbb{R}$ and such that $g(0) \neq 0$. Knowing that 0 is a stationary point of g and that $g''(0) > 0$, show that 0 is a local minimum of $h = f \circ g$.

(7) Let $f(x) = \sqrt{x}$.
  (a) **(1.5)** Find the linear approximation to f around 1 and use it to obtain an approximation of $\sqrt{1.1}$.
  (b) **(1.5)** Compute $\int_0^{\pi^2} \frac{\cos f(x)}{f(x)}\,dx$.
  (c) **(1.5)** Determine

$$\lim_{x \to 0^+} \frac{\sin f(x)}{f(x)}.$$

(8) **(1.5)** Let $f\colon \mathbb{R} \to \mathbb{R}^+$ a differentiable function on its domain and $p \in \mathbb{R}$. Show that

$$El_x[f(x)]^p = p\,El_x f(x).$$

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM

January 4, 2011

2 hours, No consultation, No calculators

1. (1.5) Find k that minimizes the distance between $\vec{u} = (-1, 1, k, 0)$ and $\vec{v} = (2, 0, 0, -7)$.

2. (1.5) For which values of $x \in \mathbb{R}$ the series $\sum_{n=0}^{+\infty} 7(\cos x)^n$ converges?

3. (1.5) Find $\lim_{x \to 0^+} x^{5x}$.

4. (1.5) Compute the following determinant:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \end{vmatrix},$$

   where $\alpha, \beta, \gamma \in \mathbb{R}$.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:

$$\begin{cases} x + 2y + z = 0 \\ -x - y + \alpha z = 3 \\ -2x - 3y + z = \beta \end{cases}$$

   a. (2.5) Classify this system depending on α and β.
   b. (0.5) Solve this system for $\alpha = 2$ and $\beta = 3$.

6. (0.5) Define rank of a matrix.

7. (0.5) Consider the vectors $\vec{a} = (0, 2, 0)$, $\vec{b} = (2, x, 0)$ and $\vec{c} = (0, 0, y)$, where $x, y \in \mathbb{R}$. Find for which values of x and y the vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

8. (1.5) Let A be an invertible matrix. Show that $(A^{-1})^{-1} = A$.

9. Consider the function $f(x) = xe^x + 3$.
   a. (0.5) Find the domain of f and discuss its continuity.
   b. (1.0) Determine the stationary points of f.
   c. (1.0) Determine the extreme points of f using the second derivative.
   d. (1.0) Discuss if the above extreme points are global.
   e. (1.0) Write the quadratic approximation of f around $x = 0$.
   f. (0.5) Find the intervals where f is invertible.
   g. (1.0) Determine the derivative of the inverse function f^{-1} at 3.

10. (1.5) Consider $f(x) = -e^{-x}$ and $g(x) = e^{-x}$. Compute the area between the graphs of f and g, with $x > 0$.

11. (1.0) Let $f\colon \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain, and $p \in \mathbb{R}$. Prove that $\text{El}_x\,[f(x)^p] = p\ \text{El}_x\,[f(x)]$.

Universidade Técnica de Lisboa – ISEG
Departamento de Matemática

Mathematics 1

1^{st} Semester 2010/2011

FINAL EXAM
January 24, 2011

2 hours, No consultation, No calculators

1. (1.5) Let $\vec{u} = (1, 2, 0, k)$ and $\vec{v} = (3, -1, -1, -1)$. Find k such that $\vec{u}$ and $\vec{v}$ are orthogonal.

2. (1.5) Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable function on its domain. Compute $El_x \dfrac{1}{f(x)}$.

3. (1.5) For each value of $\beta \in \mathbb{R}$ find
$$\lim_{x\to 1} \frac{x^3 - 3\beta x + 3\beta - 1}{(x-1)^2}.$$

4. (1.5) Decide if the function $f(x) = \sqrt{x + \alpha}$, with $\alpha \in \mathbb{R}$, is linear or non-linear.

5. Given $\alpha, \beta \in \mathbb{R}$, consider the linear system of equations:
$$\begin{cases} x + y + z = 1 \\ 2x + 5z = 1 \\ x - y + \alpha z = \beta \end{cases}$$
   a. (2.0) Classify this system depending on α and β.
   b. (1.0) Solve this system for $\alpha = 0$ and $\beta = 0$ using Cramer's rule.

6. (1.0) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

7. (1.0) Let A, B, C, X be $n \times n$ matrices and let I be the $n \times n$ identity matrix. Assuming that C and $A - B$ are invertible, solve the equation $AXC = BXC + I$ with respect to X.

8. (1.5) Consider the series

$$\sum_{n=0}^{+\infty} \left(\frac{3x+2}{3} \right)^n .$$

Discuss for which values of x the series converges, and compute its sum whenever possible.

9. Consider the function $g(x) = e^{(x^2+1)}$.
   a. (0.5) Find the domain of g and discuss its continuity.
   b. (1.0) Determine the stationary points of g.
   c. (1.0) Determine the extreme points of g using the second derivative.
   d. (0.5) Study the concavity of g.
   e. (0.5) Discuss if the above extreme points are global.
   f. (1.5) Compute $\displaystyle\int_0^1 xg(x)\,dx$.

10. Recall that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

   a. (1.5) Estimate the value of $\arctan(0.1)$.
   b. (1.0) Give an upper bound for the approximation error.

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1 2011/2012

FINAL EXAM January 3, 2012

2 hours, no consultation, no calculators

1. (1.5) Let A, B and C be $n \times n$ invertible matrices. Find the solution X of the equation $A^{-1}XB + B = C$.

2. (1.5) Consider a differentiable function f on $\mathbb{R}$ such that $f(1) = 1$ and $f'(1) = 3$. Determine the elasticity of $h(x) = f(e^{\cos x})$ with respect to x at $\pi/2$.

3. (1.5) Compute the area between the functions $\sqrt{x}$ and $-\sqrt{x}$ when $x \in [0, 1]$.

4. (1.5) Let $\vec{u} = (1, 0, 2, \pi)$ and $\vec{v} = (2, 0, k, 1)$ with $k \in \mathbb{R}$. For which value of k are these vectors orthogonal?

5. When the market is in equilibrium, the production x of wheat and its demand y, and the production z of bread and its demand t satisfy the following equations:

$$\begin{cases} x + 4y + 4t &= 350 \\ x - 4y &= -50 \\ y + z + 4t &= 250 \\ z - \frac{3}{2}t &= -50, \end{cases}$$

where all the amounts are expressed in tons.
   a. (1.5) Write this system using matrices and classify it by studying the rank.
   b. (1.0) Determine the production of wheat and bread that satisfy this equilibrium.

6. (1.5) Find for which values of $k \in \mathbb{R}$ the vectors $\vec{u} = (1, k, 1, k)$, $\vec{v} = (k, 1, k, 1)$ and $\vec{w} = (0, 1, 1, 1)$ and $\vec{t} = (0, 0, 0, 2)$ are linearly independent.

7. (1.5) Let A be an invertible $n \times n$ matrix and B be a matrix constructed by multiplying a row (or a column) of A by $\lambda \in \mathbb{R}$. Show that $|B| = \lambda\,|A|$.

8. (2.0) An investor has two investment projects:
   - the first one consists on a bond with yearly payments that correspond after ten years to the value
   $$\sum_{t=1}^{10} \frac{1200}{2^t}.$$
   - the second one generates a profit in continuous time, and after 10 years it yields
   $$\int_0^{10} 120\,t\,e^{t/2}\,dt.$$

   Which of the two investment projects is more profitable after 10 years?

9. A traditional ice cream shop produces each day up to 80 Kg of ice cream. According to the specific characteristics of this product, the profit function of the shop is given by:
   $$L(x) = \begin{cases} x^3 + 3x - 10, & 0 \le x \le 3 \\ -x^2 + 10x + 5, & 3 < x \le 8, \end{cases}$$
   where x represents the tens of Kg of ice cream.
   a. (1.0) Discuss the continuity of L.
   b. (1.0) Study the differentiability of L at $x = 3$ using the definition. Write the function L'.
   c. (1.5) Determine the stationary points of L and classify them using the second derivative.
   d. (1.5) Compute the amount of ice cream (in tens of Kg) that maximizes and that minimizes the profit of the shop.

10. (1.5) Let
   $$f(x) = \int_0^x e^{t^2}\,dt.$$
   Write the quadratic approximation of f around $x = 1$. *Note:* It is not necessary to find $f(1)$ explicitly.

Universidade Técnica de Lisboa – ISEG

Departamento de Matemática

Mathematics 1 2011/2012

FINAL EXAM January 24, 2012

2 hours, no consultation, no calculators

1. (1.5) Let A be an invertible matrix and A' its transpose. The determinant of $A\,A'$ is:

a) negative b) zero

c) positive d) negative or positive

2. (1.5) Consider $g(x) = \arctan(f(x))$, where f is a differentiable function f on $\mathbb{R}$ such that $f(-1) = 1$ and $f'(-1) = 3$ and $\arctan(1) = \frac{\pi}{4}$. The equation for the tangent line to the graph of g at $x = -1$ is:

a) $y - \frac{\pi}{4} = \frac{3}{2}(x+1)$ b) $y + \frac{\pi}{4} = \frac{3}{2}(x+1)$

c) $y + \frac{\pi}{4} = -\frac{3}{2}(x+1)$ d) None of the above

3. (1.5) The sum of the series $\sum\limits_{n=1}^{+\infty}(2-3x)^{n-1}$ is:

a) $\frac{1}{1-3x}$ if $x \in]-1,-1/3[$ b) $\frac{1}{3x-1}$ if $x \in]1/3,1[$

c) $\frac{1}{3x-1}$ if $x \in]-1/3,1/3[$ d) None of the above

4. (1.5) The value of the integral $\int\limits_0^1 \frac{x^2}{5-x^3}\,dx$ is:

a) $\ln 4 - \ln 5$ b) $-3\ln\frac{4}{5}$

c) $-\frac{1}{3}\ln\frac{4}{5}$ d) None of the above

5. The productivity indices x, y, z of three competing companies are related to the reference interest rate t by the following equations:

$$\begin{cases} x - y + t &= 0 \\ -x - y + z - \alpha t &= \beta \\ x + y - z + 2t &= 2 \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ are parameters associated to fiscal incentives. The entity that regulates α and β does not want to choose values that make the system impossible, but instead to find those that allow the largest number of solutions.

a) (1.5) Write the system using matrices and find α and β that give no solutions.

b) (0.5) Find α and β that give infinite solutions.

c) (1.0) Find the solutions for $\alpha = 2$ and $\beta = -2$.

6. (1.0) Let $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^4$ be linearly independent vectors and $\vec{v_4} = \lambda_1\vec{v_1} + \lambda_2\vec{v_2} + \lambda_3\vec{v_3}$, with $\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\}$ and $\lambda_3 \in \mathbb{R}$. Are the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ linearly independent?

7. (1.5) Let $f(x) = \int_0^{\sqrt{x}} t\cos(t)\,dt$ with $x \geq 0$. Compute $El_x f(x)$.

8. The demand p of a given asset is given by the function $p(x) = e^{-(x-1)^2}$ where $x \geq 0$ is the price of the asset.

a) (1.0) Determine the stationary points of the function p and classify them using the second derivative.

b) (1.0) What is the price that maximizes the demand?

c) (1.0) Study the concavity of p, stating the inflexion points.

d) (1.5) Find the linear approximation of p around the origin and estimate the error for $x = 0.1$.

e) (1.5) Let $g(x) = (x-1)\,p(x)$. Compute the area between the graph of g and the x-axis for $x \in [0, +\infty[$.

9. (1.0) Using the definition, prove that $\lim_{x\to 1}(5x+3) = 8$.

10. Consider a real valued function

a) (0.5) Write the definitions of strictly increasing function and injective function.

b) (1.0) Using only the definitions of the previous question, show that if a function is strictly increasing, then it is injective.