Week 2: Chap. 1 – Vectors

1 Direct applications

1.1. Consider the vectors of $\mathbb{R}^2$: $\vec{u} = (1, 2)$ and $\vec{v} = (-1, 1)$. Sketch them in the plane and determine geometrically: a) $\vec{u} + \vec{v}$ b) $\vec{u} - \vec{v}$ c) $-\vec{u} + 3\vec{v}$ d) $||\vec{u}||$ e) $d(\vec{u}, \vec{v})$.

1.2. Solve analytically the previous exercise and compare the results.

1.3. Consider the vectors of $\mathbb{R}^3$: $\vec{u} = (a, 1 + a, 2a), \vec{v} = (1, 1, 3)$ and $\vec{w} = (2, 1, 0)$. Determine the value of $a \in \mathbb{R}$ so that the vector $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$.

1.4. The vectors $\vec{u} = (-1,\ -1,\ -a,\ -1)$ and $\vec{v} = (a + 2,\ a,\ a,\ a)$, with $a \in \mathbb{R}$, are orthogonal if and only if:

a) $a = 2$ b) $a = 0$ c) $a = -2$ or $a = -1$ d) $a = 1$.

1.5. Compute the distance $d(\vec{u}, \vec{v})$ for the vectors in exercise 1.4 (with $a \in \mathbb{R}$).

2 Definitions and proofs

2.1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Show the following properties of the inner product:
a) $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
b) $(\lambda\vec{u}).\vec{v} = \vec{u}.(\lambda\vec{v}) = \lambda(\vec{u}.\vec{v})$

2.2. Find the distance $d(\vec{u}, \vec{v})$ between the vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$.

2.3. Proof that $||\vec{u}|| > 0$ for any $\vec{u} \in \mathbb{R}^n \backslash \{\vec{0}\}$.

2.4. Define linear combination of vectors.

2.5. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be linearly independent. Show that $\vec{u}+\vec{v}$, $\vec{u}+\vec{w}$ and $\vec{v}+\vec{w}$ are also linearly independent.

3 Problems and modelling

3.1. Assume the following economic data, without units:

Country	Productivity	Competition	Economic growth
Portugal	3	2	-1
Canada	8	5	0
Thailand	1	1	-3

a) Which country is closer to Portugal in all three indices?
b) In this model, the portuguese data depends linearly on the others?

3.2. Assume the following grades and weights:

Course	Weight	John's grades	Leonor's grades
Mathematics	3/10	?	15
Accounting	3/10	18	12
Law	3/10	10	14
English	1/10	16	15

a) Compute the average grade of the above students using the inner product of vectors.
b) What is the grade that John needs to obtain in Maths so that he has the same average grade as Leonor?

4 Additional exercises

4.1. Book (K. Sydsaeter & P.J. Hammond, *Essential Mathematics for Economic Analysis*, Prentice Hall, 2008):
Section 15.7: 1 to 8;
Section 15.8: 1 to 6.

4.2. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show the triangular inequality: $||\vec{u} + \vec{v}\,|| \leq ||\vec{u}\,|| + ||\vec{v}\,||$.
(Hint: Decompose $||\vec{u} + \vec{v}\,||^2$ and use the Cauchy-Schwarz inequality.)

1 Direct applications

1.1. Book (K. Sydsaeter & P.J. Hammond, *Essential Mathematics for Economic Analysis*, Prentice Hall, 2008):

Section 15.2: 1, 2 and 4;
Section 15.3: 1, 3 and 4;
Secção 15.4: Exercícios 1, 2 e 4;
Secção 15.5: Exercícios 1 a 4.

1.2. Find an example of a diagonal matrix D and a vector $\vec{v}$ with the same dimension, and compute $D\vec{v}$.

1.3. Give an example of an upper triangular matrix S and find its transpose.

1.4. Give two vectors $\vec{u}$ and $\vec{v}$ with the same dimension, and a linear combination of them.

1.5. Determine the rank of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 8 & 16 \end{bmatrix},\ B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}.$$

2 Definitions and proofs

2.1. Let A and B be matrices with dimension $k \times p$, and $\alpha, \beta \in \mathbb{R}$. Show that:
a) $A + B = B + A$ b) $(\alpha + \beta)A = \alpha A + \beta A$ c) $\alpha(A + B) = \alpha A + \alpha B$.

2.2. Let I be the identity matrix of dimension n and $k \in \mathbb{N}$. Prove that $I^k = I$.

2.3. Consider a matrix A with dimension $k \times p$, a matrix B with dimension $p \times \ell$ and $\lambda \in \mathbb{R}$. Prove that:
a) $(\lambda A)' = \lambda A'$ b) $(AB)' = B'A'$.

2.4. Let a matrix A with dimension $m \times n$. Show that if $n = 1$, $A'A = 0 \Rightarrow A = \mathbf{0}$.

2.5. Consider two commutable matrices A and B (i.e. $AB = BA$) and C a matrix such that $C = 3A^2 - 5A - I$, where I is the identity. Show that C and B commute.

3 Problems and modelling

3.1. Consider the matrix $R(\theta) = \begin{bmatrix} \cos\theta & \text{-}\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and the vector $\vec{e_x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
a) Represent on the unit circle and find the values of $\sin\theta$ and $\cos\theta$ for the angles $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2, \pi, 3\pi/2$.
b) Compute $R(\theta)\vec{e_x}$ and sketch the result, concluding that $R(\theta)$ represents the rotation of $\vec{e_x}$ by θ around the origin.
c) Verify that $[R(\theta)]^2 = R(2\theta)$ using the identities: $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
d) Interpret geometrically the previous result.

3.2. Three companies presented the following results (in million euros) in 2008:

Company	Quarter 1	Quarter 2	Quarter 3	Quarter 4
1	5	2	-1	2
2	2	8	0	5
3	1	3	-1	2

In 2009 the results were:

Company	Quarter 1	Quarter 2	Quarter 3	Quarter 4
1	2	7	3	5
2	4	4	6	6
3	-1	-1	-1	0

a) Determine, for each company, in each quarter, the changes between 2008 and 2009.
b) Determine, for each company, in each quarter, the average result of the two years.

3.3. Consider the set of vectors $\{(1,0,0),(0,1,0),(-4,2,-8)\}$.
a) Determine by the definition if it is a set of linearly independent vectors.
b) Define rank of a matrix and determine the previous result by studying the rank.

4 Additional exercises

4.1. Determine the rank of the matrices:

$$A=\begin{bmatrix}1&3&0&0\\2&4&0&-1\\1&-1&2&2\end{bmatrix},\ B=\begin{bmatrix}2&1&3&7\\-1&4&3&1\\3&2&5&11\end{bmatrix},\ C=\begin{bmatrix}1&-2&-1&1\\2&1&1&2\\-1&1&-1&-3\\-2&-5&-2&0\end{bmatrix}.$$

4.2. Book (K. Sydsaeter & P.J. Hammond, *Essential Mathematics for Economic Analysis*, Prentice Hall, 2008):
15.2: 3;
15.3: 2, 5;
15.4: 3, 6, 7;
15.5: 5, 7.

Week 4: Chap. 3 – Systems of linear equations $k \times n$

1 Direct applications

1.1. Consider the following equations/systems: $i)\ x+3=1$ $\quad ii)\ x^2=4$ $\quad iii)\ \begin{cases} x+3=1 \\ x^2=4 \end{cases}$.

a) Identify the non-linear equation.
b) Solve these equations analytically.
c) Solve these equations graphically.

1.2. Determine the set of solutions of the equation $x+y=1$ and classify it.

1.3. Solve and classify the system of equations: $\begin{cases} x+y=1 \\ x+y=-1 \end{cases}$. Verify the result graphically.

1.4. Discuss the existence of solutions for the following systems, finding — whenever possible — the number of degrees of freedom and the solutions:

a) $\begin{cases} -2x-3y+z=3 \\ 4x+6y-2z=1 \end{cases}$ b) $\begin{cases} x-y+2z+w=1 \\ 2x+y-z+3w=3 \\ x+5y-8z+w=1 \\ 4x+5y-7z+7w=7 \end{cases}$

c) $\begin{cases} x-y+z=0 \\ x+2y-z=0 \\ 2x+y+3z=0 \end{cases}$ d) $\begin{cases} x+y+z+w=0 \\ x+3y+2z+4w=0 \\ 2x+y-w=0 \end{cases}$.

1.5. Determine, for the system $\begin{cases} y+az=0 \\ x+by=0 \\ by+az=1 \end{cases}$, depending on the real parameters a and b:

a) the number of independent equations;
b) the number of useless equations;
c) the number of incompatible equations.

1.6. Classify the system of equations depending on the parameter $a \in \mathbb{R}$: $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ x-y-z=a \end{cases}$.

2 Definitions and proofs

2.1. Define equation, system of equations and degree of freedom of a system of equations.

2.2. Let $a \in \mathbb{R}\backslash\{0\}$ and $b, c \in \mathbb{R}$ be constants. Show that $ax+b=c \Leftrightarrow x=\dfrac{1}{a}(c-b)$.

2.3. Consider three vectors with dimension 4: $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$.
a) Show that the vector equation $\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\lambda_3\vec{v}_3=\vec{0}$, with variables λ_i, correspond to a system of 4 linear equations with 3 variables.
b) Is this system possible? Why?
c) Generalise the previous results for ℓ vectors with dimension p.

2.4. Decide if the statements below are true or false:

a) A linear system of equations with the same number of equations and of variables has a unique solution.

b) A linear system of equations with the same number of equations and of variables has at least one solution.

c) A linear system of equations with more equations than variables can have an infinite number of solutions.

d) A linear system of equations with less equations than variables can have no solutions.

3 Problems and modelling

3.1. An auto factory uses 3 different types of steel for the production of each of its 3 car models: A, B and C. Each model needs the following amount of steel (in tons):

Steel type \ Car	Model A	Model B	Model C
1	2	3	4
2	1	1	2
3	3	2	1

Find the amount of cars that can be produced using 29, 13 e 16 tons of type 1, 2 and 3 steel, respectively.

3.2. Consider: $\left\{\begin{array}{rl} x + 2y - \alpha z &= 1 \\ 2x - \quad y - \quad z &= \beta \\ 9x - 2y + \quad z &= -1 \end{array}\right.$, com $\alpha, \beta \in \mathbb{R}$.

a) Classify the system depending on α and β.
b) Solve it for $\alpha = \beta = 0$.
c) Show that the distance between the solution in *b)* and the vector $\left(-\frac{24}{25}, -\frac{38}{25}, -\frac{2}{5}\right)$ is $\sqrt{5}$.

4 Additional exercises

4.1. Classify the systems with respect to the parameters a and b:

a) $\left\{\begin{array}{l} x + y + z = 3 \\ x - y + z = 1 \\ 2x - 2y + az = 2 \end{array}\right.$ b) $\left\{\begin{array}{l} x + y + z = 1 \\ x - y + 2z = a \\ 2x + bz = 2 \end{array}\right.$.

4.2. Let $A\vec{x} = \vec{b}$ a system with 4 equations and 5 variables. Knowing that it has 2 d.o.f., find the rank of the matrix A:

a) 2 b) 3 c) 4 d) 1

4.3. Consider $A = \begin{bmatrix} 1 & 1 & a & 1 \\ 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 3 & 1 & b \end{bmatrix}$, with $a, b \in \mathbb{R}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ with $x_i \in \mathbb{R}$, $i = 1, \ldots, 4$.

a) Discuss the rank of the matrix A as a function of a and b.
b) Find a and b such that $A\vec{x} = \vec{0}$ has a unique solution.

4.4. Let $\left\{\begin{array}{rl} x+\ \ y+\ \ \ z &= 1 \\ x+2cy+2cz &= 1 \\ 2x+\ \ y+\ \ cz &= b \end{array}\right.$.

Find the correct answer:

a) If $c \neq 1$ and $b \neq 1$ the system has infinite solutions

b) If $c = 1$ or $c = \frac{1}{2}$, $\forall\, b \in \mathbb{R}$ the system has a unique solution.

c) If $c = 1$ and $b \neq 2$ the system has no solutions.

d) If $c \neq \frac{1}{2}$, $\forall\, b \in \mathbb{R}$ the system has infinite solutions.

4.5. Book:

15.1: 1, 3, 5 e 6;

15.6: 1 a 4.

Maths I

Week 5: Chap. 4 – Systems of linear equations $n \times n$

1 Direct application

1.1. Use a determinant to compute the area of the paralelogram defined by the vectors:
a) $\vec{u} = (3, 0)$ and $\vec{v} = (2, 6)$;
b) $\vec{u} = (\alpha, \alpha)$ and $\vec{v} = (\beta, \beta)$, with $\alpha, \beta \in \mathbb{R}$.

1.2. Use a determinant to compute the volume of the rectangular prism definined bu the vectors:
a) $\vec{u} = (3, 0, 0)$, $\vec{v} = (2, 6, 0)$ and $\vec{w} = (0, 0, 2)$;
b) $\vec{u} = (1, 0, 0)$, $\vec{v} = (0, 1, 0)$ and $\vec{w} = (0, 0, 1)$;
c) $\vec{u} = (2, 0, 0)$, $\vec{v} = (0, 2, 0)$ and $\vec{w} = (0, 0, 2)$;
d) $\vec{u} = (1, 0, 0, 0)$, $\vec{v} = (0, 1, 0, 0)$, $\vec{w} = (0, 0, 1, 0)$ and $\vec{s} = (0, 0, 1, 0)$.

1.3. Use Cramer's rule to solve the following systems os equations, and then check the solutions obtained:

$$\text{a)} \begin{cases} x_1 - x_2 + x_3 = 2 \\ x_1 + x_2 - x_3 = 0 \\ -x_1 - x_2 - x_3 = -6 \end{cases} \qquad \text{b)} \begin{cases} x_1 - x_2 & = 0 \\ x_1 + 3x_2 + 2x_3 & = 0 \\ x_1 + 2x_2 + x_3 & = 0 \end{cases} \qquad \text{c)} \begin{cases} x + 3y - 2z = 1 \\ 3x - 2y + 5z = 14 \\ 2x - 5y + 3z = 1 \end{cases}.$$

1.4. Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, such that $|A| = k$ with $k \in \mathbb{R}$. Find the value of:

$$\text{a)} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}; \qquad \text{b)} \begin{vmatrix} 2a_{11} & a_{12} & a_{13} \\ 2a_{21} & a_{22} & a_{23} \\ 2a_{31} & a_{32} & a_{33} \end{vmatrix}; \qquad \text{c)} \left| 3 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right|;$$

$$\text{d)} \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}; \qquad \text{e)} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \qquad \text{f)} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{vmatrix}.$$

1.5. The value of $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & x & y \\ 2 & 1 & y & x \end{vmatrix}$ is equal to:

a) $-x^2 + y^2 + x + y$ b) $x^2 + y^2 + x - y$
c) $x^2 - y^2 - x - y$ d) None of the above.

1.6. Book:
16.5: 2.

2 Definitions and proofs

2.1. Show that if a square matrix has a zero row or column, then its determinant is zero.

2.2. Let M be an upper triangular matrix of order 4.
a) Prove that its determinant is the product of the coefficients on the diagonal.
b) Show that the previous result is also true for a lower triangular matrixof order 4.

2.3. A square matrix M is said to be *orthogonal* if $M'M = I$, where I is the identity matrix. Prove that:
a) The product of two orthogonal matrices $n \times n$ is again an orthogonal matrix.
b) The determinant of an orthogonal matrix is always $+1$ or -1.

2.4. Without computing the determinants, show that: $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$.

3 Problems and modelling

3.1. A company in the food business produces daily wheat flour, rye flour and corn flour. The total production is k tons per day. Knowing that the total daily production of wheat and rye flours is three times the corn flour production, and that the total daily production of wheat and corn flours is twice the rye flour production, use Cramer's rule to find the wheat flour daily production.

3.2. Consider: $\begin{cases} x+z & = & a \\ y+az & = & 0 \\ x+(a+1)\,z & = & a+b \end{cases}$, with $a, b \in \mathbb{R}$.
a) Use Cramer's rule to find a and b so that this system has a unique solution. What can we say about the system for other values of a and b?
b) Determine the solutions of the system when $a = 1$ and $b = 1$.

3.3. Book:
16.1: 6 e 7.

4 Additional exercises

4.1. Let A and B square matrices of order n such that $|A| = k$ and $|B| = q$, with $kq \neq 0$. The determinant of $C = qkAB$ is:

a) $(qk)^n$ b) $(qk)^{n+1}$ c) $(qk)^2$ d) None of the above.

4.2. Consider: $\begin{bmatrix} 1 & k & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $k \in \mathbb{R}$.
a) Determine k such that the distance between columns 1 and 2 of the matrix containing the systems coefficients is $2\sqrt{2}$.
b) Consider $k = 0$ and check the triangular inequality for the distances between the columns of the matrix.
c) Consider $k = 1$ and use Cramers's rule to find y.

4.3. Book:
16.3: 1 e 2;
16.4: 1 a 8;
16.5: 1.

Week 6, Chap. 4 (cont.) – Inverse matrix and Chap. 5 – Sequences and series

1 Direct applications

1.1. Let M, P, Q, X be square matrices, with M invertible. The solution of the equation $XM + P = Q$ is $X = M^{-1}(Q - P)$: true or false?

1.2. Determine the inverse of the following matrices:

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$, with $x, y, z \in \mathbb{R}\backslash\{0\}$.

1.3. Prove that the inverse of $\begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{bmatrix}$.

2 Definitions and proofs

2.1. Let A be a matrix $n \times n$. Show that if A is invertible, then $|A| \neq 0$.

2.2. Prove that the inverse of a matrix, whenever exists, is unique.

2.3. Let A and B be invertible matrices and $\lambda \in \mathbb{R}\backslash\{0\}$. Show that the following properties hold:
a) $(A^{-1})^{-1} = A$;
b) $(AB)^{-1} = B^{-1}A^{-1}$;
c) $(\lambda A)^{-1} = \dfrac{1}{\lambda}A^{-1}$.

3 Problems and modelling

3.1. Show that the vectors $\vec{a} = (1, 1, 0)$, $\vec{b} = (0, 1, 1)$ and $\vec{c} = (1, 0, 1)$ are linearly independent:
a) using the definition of linear independence;
b) studying the rank of a matrix;
c) computing the determinant of a matrix;
d) determining if a matrix is invertible.

3.2. Recall problem 3.1 of week 2/3: show that the matrix $R(-\theta)$ is the inverse of $R(\theta)$. Discuss this result from a geometrical point of view.

3.3. Recall problem 3.1 of week 5: write the respective system of equations as $A\vec{x} = \vec{b}$ and determine the solutions $\vec{x}$ by the computation of A^{-1}.

4 Additional exercises

4.1. Let A be a symmetric and invertible matrix with dimension n such that $X'A = B + A$. Then:

a) $X = B'A^{-1} + I$ b) $X = A^{-1}B' + I$ c) $X = \left(\frac{B+A}{A}\right)'$

d) None of the above.

4.2. Consider A, B and C square matrices with dimension n. Knowing that: $|A+B| = 3$, $|C| = 2$ and $(AX + BX)' = C$, obtain X (depending on A, B and C) and compute its determinant.

4.3. Consider the matrix $A = \begin{bmatrix} 1 & x & 0 & 1 \\ x & 1 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 1 & x \end{bmatrix}$.

a) Show that its determinant is $-\left(1 - x^2\right)^2$.

b) Find the values of x for which the matrix A does not have an inverse.

4.4. Book:

16.6: 2, 6;

16.7: 2, 5.

5 Direct application

5.1. Book:

10.4: 2 - 4.

5.2. Determine if the following series are convergent. If that is the case, find their values:

a) $\sum\limits_{n\geq 0} \left(\frac{1}{2}\right)^n$ b) $\sum\limits_{n\geq 1} 3^n$ c) $\sum\limits_{n\geq 0} \left(\frac{2}{3}\right)^{n+2}$ d) $\sum\limits_{n\geq 3} \left(\frac{1}{4}\right)^{2n}$ e) $\sum\limits_{n\geq 2} 5^{-n}$.

6 Definitions and proofs

6.1. Define:

a) Function

b) Real-valued function

c) Sequence

d) Series.

6.2. Prove that $\sum\limits_{\ell=0}^{n-1} ak^\ell = a\dfrac{k^n - 1}{k - 1}$, where $a \in \mathbb{R}$, $k \in \mathbb{R}\backslash\{1\}$, and $n \in \mathbb{N}$.

7 Problems and modelling

7.1. Determine the values of $x \in \mathbb{R}$ for which the following series converge and compute their values:

a) $\sum\limits_{n=1}^{\infty} \left(1 - x^2\right)^n$ b) $4x^2 + 16x^4 + 64x^6 + ...$

7.2. Use the theory of the geometric series to write the following numbers as irreducible fractions:

a) $0,999...$ b) $1,666...$ c) $0,1212...$

7.3. Book:

10.4: 6, 7.

8 Additional exercises

8.1. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{an^2 + n}{n^2 - 1}$, with $a \in \mathbb{R}$. Find the correct answer:

a) if $a \neq 0$, then the series diverges   b) if $a \neq 0$, then the series converges
c) the series is convergent, $\forall a \in \mathbb{R}$ d) the series is convergent for $a = 1$.

8.2. Compute $\sum\limits_{n=0}^{\infty} \left[(-\frac{1}{2})^n + (\frac{1}{2})^n\right]$.

8.3. Find the values of $x \in \mathbb{R}$ for which the following series converge and compute their values:

a) $\displaystyle\sum_{n\geq 0} (3x - 4)^n$ b) $\displaystyle\sum_{n\geq 0} \left(\frac{x-1}{x+1}\right)^n$ c) $\displaystyle\sum_{n\geq 0} \frac{2^n}{(x+1)^{2n}}$

d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x-3}{2}\right)^n$ e) $\displaystyle\sum_{n=0}^{\infty} (1 - |x|)^n$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n}{(x+1)^{3n}}$ g) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2^{n-1}}$.

8.4. Book:

10.4: 5, 8.

Week 7: Chap. 5 – Real functions, and Chap. 6 – Variations

1 Direct applications

1.1. Book:
6.5: 1, 4.

1.2. Sketch the graph of the following functions:

a) $-x^2$ b) $-\sqrt{x}$ c) e^x d) $\ln x$ e) $\dfrac{1}{x}$ f) $\sin x$ g) $\cos x$ h) $\tan x$

i) $ax + b$ with $a, b \in \mathbb{R}$ j) $|x+5|$ k) $\ln(x-5)$ ℓ) an odd function.

1.3. Compute the derivative with respect to x of the functions in questions *a)* to *i)* in 1.2.

1.4. For which values of a and b the function $f(x) = \begin{cases} ax - 2 & \text{se } x \leq 1 \\ b - 2x^2 & \text{se } x > 1 \end{cases}$ is continuous?

1.5. Let $f(x) = e^x$, $g(x) = x^n$ with $n \in \mathbb{Z}$, and $h(x) = \sin x$. Compute:

$$a) \frac{d}{dx}\left[f(x) + g(x) + h(x)\right] \quad b) \frac{d}{dx}\left[5f(x) + 2g(x)\right] \quad c) \frac{d}{dx}\left[g(x)h(x)\right]$$

$$d) \frac{d}{dx}\left[f(x)g(x)h(x)\right] \quad e) \frac{d}{dx}\left[\frac{h(x)}{f(x)}\right] \quad f) \frac{d}{dx}\left[\frac{g(x)h(x)}{f(x)}\right].$$

1.6. Let $f(x) = \sqrt{x}$.

a) Find the domain of f and discuss its continuity and differentiability.

b) Compute: $\dfrac{df(x)}{dx}$, $\dfrac{d^2f(x)}{dx^2}$ and $\dfrac{d^3f(x)}{dx^3}$.

2 Definitions and proofs

2.1. Prove by the definition that: $\lim_{x \to 2} 3x + 1 = 7$.

2.2. Consider the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$. Show that if f and g are continuous in $a \in \mathbb{R}$, then $(f+g)$ is also continuous in a.

2.3. Let $f(x) = x^2$. Prove by the definition that: $\dfrac{df(x)}{dx} = 2x$.

2.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Show that $\dfrac{f(a) - f(x)}{a - x} = \dfrac{f(x+h) - f(x)}{h}$, with $h = a - x$.

2.5. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions and $k \in \mathbb{R}$. Show that:

a) $\dfrac{d}{dx}\left[f(x) + g(x)\right] = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$.

b) $\dfrac{d}{dx}\left[kf(x)\right] = k\dfrac{df(x)}{dx}$.

3 Problems and modelling

3.1. The stock price for the following companies is given with respect to time t by:

- Company A: $2t^2 + 4t$
- Company B: $3t^2 + t$
- Company C: $\frac{2t}{t^2+1}$.

a) At $t = 1$ which company has the fastest growing stock price?
b) In what period of time the stock price of C is growing?

3.2. Study the domain, continuity and diferentiability of:

a) $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$ b) $g(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{se } x < 0 \\ \ln\left(1 + x^2\right) & \text{se } x \geq 0 \end{cases}$.

3.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Solve the equation: $\dfrac{df(x)}{dx} = f(x)$.

3.4. Book:
7.8: 4;
6.7: 8;
6.9: 9, 10.

4 Additional exercises

4.1. Determine the domain of:

a) $f(x) = \dfrac{1}{x+3}$ b) $g(x) = \dfrac{x}{x^2+1}$ c) $h(x) = \ln(3 - 2x)$

d) $i(x) = \sqrt{x^2 - 25}$ e) $j(x) = \dfrac{1}{\sqrt{x^2 - 4}}$ f) $k(x) = \ln(\ln x)$

g) $l(x) = \dfrac{1}{\ln(1 - |x - 1|)}$ h) $m(x) = \dfrac{\ln(4 - x^2)}{\sqrt{e^x - 1}}$.

4.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a two times differentiable function. Solve the equation: $\dfrac{d^2 f(x)}{dx^2} = -f(x)$.

4.3. Book:
7.8: 2, 3, 5;
7.9: 1 to 3;
6.5: 5;
6.7: 6, 7;
6.9: 1, 3, 7.

Week 8: Chap. 6 – Diferentials, Composite functions, De l'Hôpital rule

1 Direct applications

1.1. Let $f(x) = e^x$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$. Determine the domain and the range of:
a) $f \circ g$ b) $f \circ h$ c) $h \circ f$ d) $h \circ f \circ g$ e) $f \circ h \circ f \circ g$

1.2. Compute the diferential of the following functions with respect to their variable:
a) $x^5 + 2x^4 + 1$ b) $-\sqrt{u}$ c) e^y d) $\ln z$ e) $\dfrac{1}{x}$ f) $\sin u$ g) $\dfrac{\sin x}{\cos x}$.

1.3. Differentiate the following functions with respect to x:

a) $\left(5x^{70} + 3x + 1\right)^2$ b) $\left(5x^2 + 3x + 1\right)^{70}$ c) $\cos\left(3x^5 - x\right)$ d) $e^{-\frac{x}{2}}$

e) $\sqrt{x-3}$ f) $\dfrac{1}{\ln x}$ g) $e^{\sin x}$ h) $x + \sqrt{x^2 - 1}$

i) $\ln(\sin x)$ j) $\ln\left(x^2 + 1\right)$ k) $\ln^4\left(\sqrt{1 - x^2}\right)$ ℓ) $e^{-\cos\left(\sqrt{x^4 + x^2 + 1}\right)}$

1.4. Find the limits:

a) $\displaystyle\lim_{x \to \frac{\pi}{3}} \frac{\sin 3x}{1 - 2\cos x}$ b) $\displaystyle\lim_{x \to \frac{\pi}{4}} \frac{e^{\sin x} - e^{\cos x}}{\sin x - \cos x}$ c) $\displaystyle\lim_{x \to \frac{1}{2}} \frac{(2x-1)^2}{e^{2x-1} - 4x^2}$

d) $\displaystyle\lim_{x \to +\infty} x \ln\left(1 + \frac{1}{x}\right)$ e) $\displaystyle\lim_{x \to -\infty} x e^{-x^2}$ f) $\displaystyle\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

2 Problems and modelling

2.1. Three plastic companies have the following production costs, depending on the oil price p:

- Company 1: $5p^3 + 2p + 1$
- Company 2: $2p^{3/2} + p$
- Company 3: $\sqrt{p} + \frac{1}{p}$.

a) Determine for each company the average rate of change for the production cost when the oil changes price from 1 €/ℓ to 4 €/ℓ.
b) Determine for each company the instantaneous rate of change of the production cost when the oil price is 1 €/ℓ.
c) Knowing that during a brief period of crisis $t \in [0, 2]$ the oil price was $p(t) = e^t$, determine which company had a faster growing production cost at $t = 1$.

2.2. Consider the function $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ e^{-kx} & \text{if } x \geq 0 \end{cases}$, with $k > 0$.

a) Find the domain of f and sketch its graph.
b) Discuss the continuity of f on its domain.
c) Discuss the differentiability of f on its domain.
d) Consider the function $g(x) = \sqrt{x}$. Discuss the continuity and differentiability of $g \circ f$ and compute its derivative where possible.

2.3. Let $h(x) = f(x \ln x)$ be differentiable on $\mathbb{R}$. Knowing that $f(0) = \sqrt{3}$ and $f'(0) = 2$, find the equation of the tangent line to the graph of h at $x = 1$.

3 Additional exercises

3.1. Differentiate the following functions with respect to x:

a) $\left(\dfrac{x-1}{x+2}\right)^2$ b) $\left(\dfrac{x^2-1}{2x}\right)^3$ c) $\sqrt{e^x+1}$ d) $e^{-\sqrt{x}}$

e) $e^{x^3} \ln\left(x^2\right)$ f) $\dfrac{3}{\sqrt{x}}$ g) $\sqrt[3]{\dfrac{3-x}{x-1}}$ h) e^{x^2}

i) $\ln\left(e^{3x} + x^2\right)$ j) $e^x \ln x$ k) $\sin(2x+1)$ ℓ) xe^x

m) $\cos x + x \cos^2\left(x^2\right)$ n) $\sin x \cos x$ o) $\tan\left(x^2+1\right)$ p) $\ln \dfrac{1+x}{1-x}$

3.2. Compute the limits:

a) $\displaystyle\lim_{x \to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$ b) $\displaystyle\lim_{x \to 0^+} \left(\frac{1}{x}\right)^{\sin x}$

3.3. Find $L = \lim\limits_{x \to 1} \frac{2x^\alpha - 2\alpha(x-1) - 2}{3x^2 - 6x + 3}$:

a) $L = -\alpha - 3$ b) $L = 0$ c) $L = \frac{\alpha^2 - \alpha}{3}$ d) L does not exist

3.4. Find $\displaystyle\lim_{x \to +\infty} \frac{\sin(5/x)}{2/x}$?

a) $\frac{5}{2}$ b) 0 c) $-\frac{5}{2}$ d) $\frac{2}{5}$.

3.5. Let f and g be differentiable functions on $\mathbb{R}$ such that $h(x) = f[g(x)]$. Knowing that $f(-1) = 2$, $f'(-1) = 1/3$, $g(3) = -1$, and $g'(3) = -4$, find the equation of the line tangent to the graph of h at $x = 3$:

a) $y = -\frac{4}{3}x + 2$ b) $y = -\frac{4}{3}x + 6$ c) $y = -4x + 2$ d) $y = -x + 5$

3.6. Book:
6.2: 5, 7.

Week 9: Chap. 6 – Elasticity, Implicit differentiation, Inverse function

1 Direct applications

1.1. Compute the elasticity in order to x of:
a) e^x b) $e^{\lambda x}$, with $\lambda \in \mathbb{R}$ c) $\dfrac{1}{x}$ d) $\cos(x^2)$.

1.2. Let $f(x) = \frac{1}{2}x^k h(x)$, with $k \in \mathbb{R}$ and h a differentiable function on its domain. Compute $El_x f(x)$.

1.3. Let f be twice differentiable on $\mathbb{R}$ such that: $2x^2 + 6xf(x) + [f(x)]^2 = 18$. Compute $\dfrac{df(x)}{dx}$ and $\dfrac{d^2 f(x)}{dx^2}$.

1.4. For each of the following functions, discuss on which intervals they are invertible, find the inverse function and sketch the graph:
a) $\ln x$ b) x^2 c) $\dfrac{1}{x}$ d) $\sin x$ e) $\tan x$.

1.5. Compute, using the derivative of the inverse function theorem, the derivative at 1 (if it exists) of the inverse functions obtained in exercise 1.4.

1.6. Let $f(x) = x^2 e^x$.
a) Determine the intervals where f has an inverse.
b) Let $g(y)$ be the inverse function of $f(x)$ and x_0 a point where $f'(x_0) \neq 0$. Find the derivative of g at $y_0 = f(x_0)$.

2 Definitions and proofs

2.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ be differentiable on $\mathbb{R}$. Given a change Δx on x, the function feels a change $\Delta f(x) = f(x + \Delta x) - f(x)$.

Prove that $\lim\limits_{\Delta x \to 0} \dfrac{\frac{\Delta f(x)}{f(x)}}{\frac{\Delta x}{x}} = \dfrac{x}{f(x)} f'(x)$.

2.2. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ be differenciable on their domain. By defining $u = g(x)$, show that $El_x (f \circ g)(x) = El_u f(u) \cdot El_x u$.

2.3. Let f be injective on $I \subseteq \mathbb{R}$, e $g = f^{-1}$. Write the equality relating f and g.

3 Problems and modelling

3.1. In a powder chocolate factory, the production cost f of chocolate, expressed in €/kg, depends on the price x of cacao, also in €/kg, as given by: $f(x) = x^2 + 3$, for $x \geq 0$. Consider a scenario where the price of cacao changed from 1 €/kg to 2 €/kg. Find the following:
a) The absolute change of the cacao price.
b) The absolute change of the production cost of chocolate.
c) The relative change of the cacao price.
d) The relative change of the production cost of chocolate.
e) The absolute rate of change of the production cost of chocolate against the increase of the cacao price.
f) The relative rate of change of the production cost of chocolate against the increase of the cacao price.
g) Consider now an infinitesimal increase dx of cacao x. Compute the absolute rate of change and the relative rate of change (elasticity) of the production cost of chocolate against the infinitesimal increase of cacao.

3.2. Imagine that the gasoline consumption c of a car depends on its speed v like: $c(v) = v^3 + 2v + 5$ (clearly, $v \geq 0$).
a) If the driver duplicates its speed, how does the gasoline consumption varies?
b) Let f the function that gives us the speed depending on the gasoline consumption: that is, $f[c(v)] = v$. Compute $f'(5)$.

3.3. Find the equation of the tangent line to the graph of f, defined implicitly by the equation $\sin[xf(x)] = f(x)$, at $(\frac{\pi}{2}, 1)$.

3.4. Let $g(x) = f[xg(x)]$ implicitly defined on $\mathbb{R}$. Knowing that $f'[g(1)] = 2$, find $g'(1)$?

3.5. Let $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $f(x) = x^x$.
a) Is it exponencial?
b) Is it polynomial?
c) Use $e^{\ln x} = x$ to compute f'.

3.6. Book:
7.7: 2, 6.

4 Additional exercises

4.1. Find $L = \lim\limits_{x \to 0^+} x^{5x}$:

a) L does not exist b) $L = 1$ c) $L = +\infty$ d) $L = 0$

4.2. Knowing that $f(x) = x^3 + 2x - 1$ admits an inverse function g and that $f(1) = 2$, find the slope of the tangent line to the graph of g at this point.

4.3. Let f be differentiable with $f(x) \neq 0$. Determine the elasticity of:

a) $x^5 f(x)$ b) $[f(x)]^{3/2}$ c) $x + \sqrt{f(x)}$ d) $\dfrac{1}{f(x)}$.

4.4. Differentiate:

a) $\tan^2(\arcsin x)$ b) $\arctan\left(x^2-1\right)$ c) $x^2 \arcsin x$ d) $\frac{1}{2}\arctan\left(e^{2x}\right)$.

4.5. Book:

7.7: 5, 9;
7.1: 1, 6, 7, 8, 10;
5.3: 3, 5, 7, 9, 11;
7.3: 1 - 3.

Week 10: Chap. 7 – Polynomial approximations, Intermediate value theorem and mean value theorem

1 Direct application

1.1. Let $f(x) = \ln x$.
a) Find the linear approximaton of f around $x = 1$.
b) Find the quadratic approximation of f around $x = 1$.
c) Sketch the graph of f and compare it with the graphs of the previous approximations.
d) Make an estimate of $\ln(1.1)$.

1.2. The quadractic approximation of $f(x) = (x+1)^5$ around $x = 1$ is given by:

a) $f(x) \simeq 80x^2 - 80x + 32$ b) $f(x) \simeq -80x^2 + 80x + 32$

c) $f(x) \simeq -80x^2 - 80x - 32$ d) $f(x) \simeq 80x^2 + 80x + 32$

1.3. Let $f(x) = \left(\frac{1}{x} - 1\right)^2$. The Taylor approximation of second degree of f around $x = 1$ is:

a) $x - 1 + (x-1)^2$ b) $x - 1 - (x-1)^2$

c) $-(x-1)^2$ d) $(x-1)^2$

1.4. Write Taylor's formula of degree n for $f(x) = e^x$ around $x = 1$, with the Lagrange's remainder. Compute the limit of the remainder when $n \to +\infty$.

1.5. Show that the equation $xe^x = \dfrac{1}{2}$ has one unique solution in $]-1, 1[$.

2 Definitions and proofs

2.1. Use the linear approximation to show that around the origin we have: $\sin x \simeq x$.

2.2. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$.
a) Define increasing function.
b) Prove that if $f'(x) \geq 0$ for $x \in]a, b[$, then f is increasing.

2.3. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable on $\mathbb{R}$ and let $p(x) = \alpha x^2 + \beta x + \gamma$, with $\alpha, \beta, \gamma \in \mathbb{R}$. Determine the coefficients α, β, γ that satisfy the following conditions for $a \in \mathbb{R}$:

$$\begin{cases} f(a) = p(a) \\ f'(a) = p'(a) \\ f''(a) = p''(a) \end{cases}.$$

3 Problems and modelling

3.1. Estimate the approximate value of $\sin(0.1)$ and its approximation error.

3.2. Let f be implicitly defined by the equation $[f(x)]^3 = x^3 f(x) + x + 1$. Knowing that $f(0) = 1$, find the linear aproximation of $f(x)$ around $x = 0$.

3.3. Consider $f(x) = e^{x-1}$.
a) Write the Taylor formula of degree n of f around 1.
b) Find an upper bound on the remainder for $x = \frac{1}{2}$ and $n = 3$.

3.4. Use the Taylor formula to compute: $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^2}$.

3.5. Let $f(x) = \sqrt{x}$. Determine the linear approximation of f around $x = 1$ and use it to get an approximation of $\sqrt{1.1}$.

4 Additional exercises

4.1. Use the Taylor formula to write the polynomial $x^3 - 2x^2 - 5x - 2$ as a sum of powers of $(x + 2)$.

4.2. Let $y = f(x)$ implicitly defined by $xy - x^2 = 2y + x$. The linear approximation of f around 4 is given by:

a) $-5x + 3$ b) $-\frac{1}{2}(x - 24)$ c) $\frac{1}{3}(x + 25)$ d) $x + 3$

4.3. Let $f(x) = (2x - a)^m$, with $m \in \mathbb{N}$. Show that the Taylor approximation of second degree of f around 0 is:

$$(-a)^m + 2m(-a)^{m-1}x + 2m(m-1)(-a)^{m-2}x^2.$$

4.4. Book:
7.4: 1 to 4, 7, 9, 10
7.5: 1, 2, 4, 5
7.6: 1, 2, 4;
7.10: 1, 2;
8.4: 6, 7.

Week 11: Chap. 8 – Extremes and concavities, Chap. 9 – Integrals and areas

1 Direct applications

1.1. Book:

8.6: 4, 5;
8.7: 5, 6.

1.2. Let $f(x) = x^3 - 4x^2 + 4x + 12$.

a) Determine the stationary points of f.
b) Determine the extreme points of f using the second derivative.
c) Find out if the extreme points are local or global.

1.3. Let $f : I \to \mathbb{R}$ such that $f(x) = \sin(x^2)$, with $I = [-\sqrt{\pi}, \sqrt{\pi}]$.

a) Determine the stationary points of f.
b) Determine the extreme points of f using the second derivative.
c) Find out if the extreme points are local or global.

1.4. Let $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$.

a) Determine the stationary points of each function.
b) Using the derivatives of order 2 or higher, determine if those points are minima, maxima or inflection points.
c) Determine the concavities of each function.

1.5. Is an inflection point always a stationary point?

2 Definitions and proofs

2.1. State the definition of increasing and decreasing functions.

2.2. State the definition of stationary point.

2.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ have a second derivative continuous on I, and a an interior point of I.
a) State the definition of inflection point of f.
b) Prove that if a is an inflection point of f, then $f''(a) = 0$.

3 Problems and modelling

3.1. A faulty freezer operates between -3ºC and +2ºC, and it has an energy consumption that varies with the temperature t as: $t^3 + \frac{3}{2}t^2 - 6t + 10$.

a) Determine the temperatures for which the energy consumption is maximum and minimum.
b) Does the function energy consumption have an inflection point?

3.2. Let $f(x) = \begin{cases} (x+2)^2, & x < -1 \\ |x|, & -1 \leq x \leq +1 \\ e^{-x+1}, & x > +1 \end{cases}$.

a) What is the domain of f?
b) Discuss the continuity and differentiability of f in its domain.
c) Determine the stationaty points of f.
d) Determine the extreme points of f, indicating if local or global.
e) Determine the extreme points of f in $[-4, -1]$.

3.3. Consider $f(x) = x \sin x$.
a) Find the Taylor polynomial of second degree of f around 0.
b) The function f has a unique stationary point in $]-1, 1[$. Determine it.
c) Classify this stationary point using the second derivative.
d) Is there any extreme points of f in $]-1, 1[$?

4 Additional exercises

4.1. Let f be the function and I the interval in exercise 1.3. Show that f has at least two inflection points in I.

4.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $a \in \mathbb{R}$ such that $f'(a) = 0$ and $f''(a) < 0$. Prove that a is a local maximum of f.

4.3. Book:
8.6: 1, 3, 6;
8.7: 2 to 4.

5 Direct applications

5.1. Compute the following anti-derivatives:

a) $\int x^2 dx$ b) $\int \sqrt{x} dx$ c) $\int e^x dx$ d) $\int \cos y dy$ e) $\int \frac{x^5}{5} dx$ f) $\int \frac{1}{2\sqrt{x}} dx$
g) $\int \frac{1}{2} dx$ h) $\int x^4 dt$ i) $\int (\sin u + x^2) dx$ j) $\int (\sin u + x^2) du$ k) $\int e^{7u} dx$ ℓ) $\int \frac{1}{2} dt$.

5.2. Compute the anti-derivative $F(x) = \int f(x) dx$:
a) such that $F(2) = 0$, for $f(x) = x^4$;
b) such that $F(0) = 1$, for $f(x) = e^x$;
c) such that $F(1) = \pi$, for $f(x) = x^{-1}$;
d) such that $F(0) = e$, for $f(x) = x^3 - 4x^2 + 4x + 12$;
e) such that $F(1) = 0$, for $f(x) = (1 - x^2)^{-\frac{1}{2}}$.

5.3. Compute the following integrals:

a) $\displaystyle\int_0^2 x^3 dx$ b) $\displaystyle\int_1^0 (-\sqrt{x})dx$ c) $\displaystyle\int_0^{\ln 1} e^{-t}dt$ d) $\displaystyle\int_{-\pi}^{\pi} \cos y dy$ e) $\displaystyle\int_0^1 \frac{1}{1+x^2}dx$

f) $\displaystyle\int_{-1}^1 (6x^5 + \frac{1}{3}x^2 - 2x + 7)dx$ g) $\displaystyle\int_2^3 (\sin u + x^{\frac{1}{3}})dx$ h) $\displaystyle\int_e^{7e} e^{7u}dx$ i) $\displaystyle\int_a^b 1dt.$

5.4. Find the area between the graph of f and the x-axis for:
a) $f(x) = x^2$ and $x \in [0, 2]$
b) $f(x) = -x^2$ and $x \in [0, 2]$
c) $f(t) = e^{-t}$ and $t \in [1, 5]$
d) $f(x) = -\sqrt{\sqrt{x}}$ and $x \in [0, 1]$
e) $f(x) = \dfrac{-x^4 - 2x^2}{x}$ and $x \in [-1, 1]$

6 Definitions and proofs

6.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function on $\mathbb{R}$, and $a, b, \lambda \in \mathbb{R}$ constants. Show that:

a) $\displaystyle\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$

b) $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

c) $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, with $a \leq c \leq b$.

6.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an odd continuous function, and $k \in \mathbb{R}$.
a) Prove that $\displaystyle\int_{-k}^{k} f(x)dx = 0.$
b) Interpret geometrically the previous result.

6.3. Let $a, b \in \mathbb{R}$ such that $a < b$, and $d(a, b)$ the distance between these two points.
a) Show that $d(a, b) = \displaystyle\int_a^b dx.$
b) Interpret geometrically the previous result.

7 Problems and modelling

7.1. An oil well has an extraction rate (measured in barrels by unit of time) that varies with time t according to: $10e^{-2t}$.

a) What is the amount of oil extracted from the well at time $t = 50$?
b) Solve the same problem for the rate 2^{-t}.

7.2. Let $f(x) = x^3 - 4x^2 + 4x$. Compute the area between the graph of f and the x-axis for $x \in [-1, 2]$.

8 Additional exercises

8.1. Book:

9.1: 1 to 9;
9.2: 1 to 6, 8.

Week 12: Chap. 9 – Integrals and areas

1 Direct applications

1.1. Study the convergence of the following improper integrals, and find their values whenever possible:

a) $\displaystyle\int_0^{+\infty} e^{-x}dx$ b) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}dx$ c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$ d) $\displaystyle\int_{-\infty}^{+\infty} u^3du$ e) $\displaystyle\int_{-\infty}^{+\infty} e^{-x}dx$.

1.2. Compute the following anti-derivatives by parts:

a) $\displaystyle\int xe^x dx$ b) $\displaystyle\int x^2\ln x dx$ c) $\displaystyle\int t\sin t dt$ d) $\displaystyle\int x^2\sin x dx$ e) $\displaystyle\int e^x\cos x dx$.

1.3. Compute the following anti-derivatives by substitution:

a) $\displaystyle\int 2x\sin(x^2)dx$ b) $\displaystyle\int e^{x^2}xdx$ c) $\displaystyle\int \frac{x^4}{x^5+7}dx$ d) $\displaystyle\int \frac{x}{1+x^4}dx$ e) $\displaystyle\int \frac{\cos x}{2\sqrt{\sin x}}dx$.

1.4. Determine: a) $\displaystyle\frac{d}{dt}\int_4^t e^{-x^2}dx$ b) $\displaystyle\frac{d}{dx}\int_x^\alpha \frac{1}{\sqrt{s^4+1}}ds$, with $\alpha\in\mathbb{R}$.

1.5. Find the area of the following subsets of $\mathbb{R}^2$:

a) $\{(x,y)\in\mathbb{R}^2\colon y\le 5, y\ge -5x+5, y\ge \ln x\}$

b) $\{(x,y)\in\mathbb{R}^2\colon 0\le y\le e^x, x\le 1\}$

c) $\{(x,y)\in\mathbb{R}^2\colon x\le y\le -x^2+2\}$.

1.6. Compute:

a) $\displaystyle\int_0^{\sin t} x^3dx$ b) $\displaystyle\int_1^e \ln x dx$ c) $\displaystyle\int_0^1 te^{-t^2}dt$ d) $\displaystyle\int_{-\pi}^{\pi} x\cos x dx$ e) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+\sin^2 x}dx$

f) $\displaystyle\int_0^1 \sqrt{3x+7}dx$ g) $\displaystyle\int_{-3}^4 |x-2|dx$ h) $\displaystyle\int_0^1 3e^{5x+1}dx$ i) $\displaystyle\int_0^\pi 2xt^3\cos t^4 dt$ j) $\displaystyle\int_1^2 \frac{1}{x}dx$.

2 Problems

2.1.

a) Find the area above the graph of $f(x)=x^2-4$ and below the x-axis.

b) Find the area between the graphs of $f(x)=x^2+x+1$ and $g(x)=2x^2+5x+4$, for $x\in[-3,0]$.

2.2. Determine the domain, the intervals of monotony and the local extreme points of:

a) $\displaystyle F(x)=\int_1^x \ln t dt$ b) $\displaystyle H(x)=\int_0^{x^2} e^{-t^2}dt$.

2.3. Consider the function $f(x) = \int_{\pi}^{x^2} e^{-2t} dt$. Write Taylor's formula with order 1 of f around $x = 0$.

2.4. Consider the function $F(x) = \int_0^x tf(t)dt$, where f is continuous and strictly positive in $\mathbb{R}$. Prove that F has a local minimum at $x = 0$.

2.5. Consider the function $f(x) = \begin{cases} a - x & \text{if } x < 0 \\ \frac{1}{b+x} & \text{if } x \geq 0 \end{cases}$, with $b > 0$.

a) Determine the values of a and b for which f is continuous.

b) Take $a = b = 1$ and consider the function $F(x) = \int_{-1}^{x} f(t)dt$. Find $F(1)$. Moreover, show that F is invertible on $(0, +\infty)$.

2.6. Let $f(x) = \int_1^{x^2+1} \left(\frac{1+t}{t}\right) dt$.

a) Find $f(-1)$.

b) Determine the equation of the tangent line to the graph of f at $x = -1$.

2.7. Consider the function with domain D_f: $f(x) = \begin{cases} -x^2 - x + 1 & \text{if } x < 0 \\ e^{2x} & \text{if } x \geq 0 \end{cases}$.

a) For which values of $x \in D_f$ the function f is differentiable? Find $f'(x)$.

b) Determine $G(x) = \int_{-1}^{x} f(t)dt$, defined in $[-1, \infty)$.

2.8. Determine the function f, twice differentiable on $\mathbb{R}$, that satisfies: $f''(x) = 2\cos x + xe^x$, $f'(0) = 2$, $f(0) = 1$

2.9. Without using anti-derivatives, compute:

a) $\lim_{x \to 0} \dfrac{\int_0^x \ln\left(t^2 + 1\right) dt}{x^3}$ b) $\lim_{x \to 0} \dfrac{\int_0^{x^2} \cos t dt}{x^2}$.

3 Additional exercises

3.1. Determine an anti-derivative of the following functions, in their respective domains:

a) x^2e^x b) $x\sqrt{x+1}$ c) $x^3\sqrt{1+x^2}$ d) $2x\cos x$ e) $\sin^2 x$

f) $\ln(2x-1)$ g) $x^2 \ln x$ h) $\arctan x$ i) $\ln^2 x$ j) $e^x \cos x$.

3.2. Determine, by substitution, an anti-derivative of:

a) $\dfrac{x}{1+x^2}$ b) $\sqrt{1-\sin^2 x}$

c) $\dfrac{e^{\frac{x}{4}}}{1+e^{\frac{x}{10}}}$, with $x = 20\ln t$ $(t > 0)$ d) $\dfrac{\cos x}{\sin^6 x}$, with $x = \arcsin t$.

3.3. Study the convergence of the improper integrals and find their values whenever possible:

a) $\int_0^{+\infty} xe^{-x^2} dx$ b) $\int_0^{+\infty} \cos x dx$ c) $\int_0^{+\infty} \frac{\arctan x}{1+x^2} dx$

d) $\int_{-\infty}^{0} \frac{1}{x^2+1} dx$ e) $\int_0^{3} \frac{1}{x-3} dx$ f) $\int_0^{2} \frac{2}{\sqrt{4-x^2}} dx.$

3.4. Find the area aobve the graph of $f(x) = \ln x$, for $x \in [0, 1]$, and below the line $y = 0$.

3.5. Book:

9.3: 4 to 6;
9.5: 2, 3;
9.6: 3;
9.7: 1, 4, 12.