Mathematics I

1st semestre - 2012/13

Exercises

Mathematical Analysis

2 Real numbers. Topological notions

2.1. Show by the induction principle:
a) $1+2+3+\ldots+n=\frac{n(n+1)}{2}$, for any $n \geq 1$,
b) Bernoulli inequality: Let $a>-1$ and $n \in \mathbb{N}$,

$$(1+a)^n \geq 1+na.$$

c) Binomial theorem:

$$(a+b)^n=\sum_{p=0}^{n}\left(\begin{array}{c} n \\ p \end{array}\right) a^{n-p}b^p, \qquad \forall_n \in \mathbb{N}, \qquad \forall_{a,b} \in \mathbb{R}.$$

Recall that $\left(\begin{array}{c} n \\ p \end{array}\right)=\dfrac{n!}{p!(n-p)!}$, which implies that

$$\left(\begin{array}{c} n+1 \\ p \end{array}\right)=\left(\begin{array}{c} n \\ p-1 \end{array}\right)+\left(\begin{array}{c} n \\ p \end{array}\right)$$

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2.2. Consider the following

$$1=1; \qquad 1-4=-(1+2); \qquad 1-4+9=1+2+3; \qquad 1-4+9-16=-(1+2+3+4).$$

Derive a law and prove it by induction.

2.3. Interpret geometrically the following sets:
a) $\{x : |x|<1\}$,
b) $\{x : |x|<0\}$,
c) $\{x : |x-a|<\epsilon\}$, where $\epsilon>0$
d) $\{x : |x|>0\}$,
e) $\{x : (x-a)(x-b)<0\}$, where $a<b$

f) $\{x : x^3 > x\}$,
g) $\{x : |x-1| \geq |x|\}$.

2.4. Solve the following equations
a) $x + 2 = \sqrt{4x+13}$
b) $|x+2| = \sqrt{4-x}$
c) $x^2 - 2|x| - 3 = 0$

2.5. Let A and B two subsets of $\mathbb{R}$ such that $A \subset B$ and suppose that A is non-empty and B is bounded from above. Justify the existence of supremum of A and B, and show that $\sup A \leq \sup B$.

2.6. Show that
a) $|x+y| \leq |x| + |y|$,
b) $||x| - |y|| \leq |x-y|$.

2.7. Solve the inequalities and determine the set of the solutions:

a) $\lvert 3-2x\rvert < 1$,	k) $\lvert 2-3x\rvert \leq 1$
b) $\lvert 1+2x\rvert \leq 1$	l) $\lvert x-3\rvert > 2$
c) $\lvert x-1\rvert > 2$	m) $\lvert 3-x^{-1}\rvert < 1$
d) $\lvert x+2\rvert \geq 5$	n) $\lvert x-4\rvert < \lvert x+2\rvert$
e) $\lvert 5-x^{-1}\rvert < 1$	o) $\lvert x^2-5\rvert \leq 2$
f) $\lvert x-5\rvert < \lvert x+1\rvert$	p) $x < x^2 - 12 < 4x$
g) $\lvert x^2-2\rvert < 1$	q) $\lvert 2x-1\rvert - x \geq 2$
h) $\lvert 2x-1\rvert = 5$	r) $\frac{x}{1+\lvert x\rvert} \leq 2$
i) $\lvert 5x-6\rvert + 3 = 10$	s) $x - 2 \geq (\lvert x\rvert - 1)^2$
j) $\lvert 2x-1\rvert = \lvert 4x+3\rvert$	t) $\left\lvert \frac{x^2-x}{1+x} \right\rvert > x$.

2.8. Determine in $\mathbb{R}$ the set of the upper bounds, of the lower bounds, the sup, the inf, the max and the min (if they exist) of the following sets:

a) $\{1, \sin 1, \sin 2\}$	e) $\{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N}\}$
b) $\{(-1)^n \frac{1}{n} : n \in \mathbb{N}\}$	f) $\{n^{(-1)^m} : m, n \in \mathbb{N}\}$
c) $\{\frac{a^n}{n!} : n \in \mathbb{N}\}$ with $a \in \mathbb{R}$ such that $\lvert a\rvert < 2$	d) $\{m + \frac{1}{n} : m, n \in \mathbb{N}\}$

2.9. Determine the interior, the exterior, the boundary, the closure, the set of the accummulation points, the set of the upper bounds, the set of the lower bounds, sup, inf, max, min (if they exist) of the following sets:

a) $A = [2,3[\cup [4,10[$,	c) $C = [0,1] \setminus \mathbb{Q}$,
b) $B =]5,7[\cup \{15\}$,	d) $D = [2,3] \cap \mathbb{Q}$.

2.10. Determine the interior, the exterior, the boundary, the closure and the set of accumulation points of the following sets:
a) $A = \{x \in \mathbb{R} : x^2 < 49\}$,
b) $B = \{x : x \text{ is irrational and } x^2 < 49\}$.

2.11. Consider the set

$$A = \left\{ x \in \mathbb{R} : x = 1 + \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$$

a) Determine the interior, exterior, boundary, closure and accumulation points of A.

b) Decide is A is open or closed.

2.12. Determine the exterior, interior, boundary and the accumulation points of:

$$A = \{x \in \mathbb{Q} : |x+3| < 5\} \cup \left\{x : x \text{ is irrational} \wedge -\sqrt{2} \leq x \leq \sqrt{13}\right\}.$$

2.13. (Exam) Let

$$A = \left\{ x \in \mathbb{R} : \left| \frac{x^2}{x-2} \right| \leq 1 \right\}$$

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$$B = \left\{ y \in \mathbb{R} : y = (-1)^n \left(2 + \frac{1}{n}\right) \wedge n \in \mathbb{N} \right\}.$$

Determine:

a) $\operatorname{int}(A \cup B)$,

b) $(A \cup B)'$.

2.14. (Exam) Let

$$A = \left\{ x \in \mathbb{R} : \left|1 - \frac{1}{x}\right| \left|\frac{1}{x} + 1\right| < \frac{1}{x^2} \right\}$$

and

$$B = \left\{ y \in \mathbb{R} : y = \frac{1+2n}{2^n} \wedge n \in \mathbb{N} \right\}$$

a) Write A using intervals.

b) Determine, if they exist, the sup and inf of $A \cap B$.

3 Numerical sequences

3.15. Compute the limit of:

a) $s_n = \frac{1}{n}$ c) $s_n = \frac{n}{n+1}$ e) $s_n = \frac{n^2}{n^4+1}$

b) $s_n = \frac{1}{n^2}$ d) $s_n = \frac{2n-1}{3n+2}$ f) $s_n = \frac{(-1)^n}{n}$

3.16. Let $(x_n) \in \mathbb{R}$, $x_n \to \infty$, $P(x) = a_0 x_n^p + \ldots + a_{p-1} x_n + a_p$ and $Q(x) = b_0 x_n^q + \ldots + b_{q-1} x_n + b_q$ two polynomials with real coefficients, $p, q \in \mathbb{N}$, $a_0 \neq 0$, $b_0 \neq 0$. Show that

a) $\lim P(x_n) = \lim a_0 x_n^p$.

b) $\lim \dfrac{P(x_n)}{Q(x_n)} = \lim \dfrac{a_0 x_n^p}{b_0 x_n^q} = \begin{cases} \dfrac{a_0}{b_0} & \text{if } p = q, \\ \infty & \text{if } p > q, \\ 0 & \text{if } p < q. \end{cases}$

3.17. Compute, if it exists, the limit of the sequences:

a) $u_n = \dfrac{1-n}{4n+3}$

b) $u_n = \dfrac{n^2+2}{3n+1}$

c) $u_n = \dfrac{3n}{4n^3+1}$

d) $u_n = \dfrac{-n^3+2}{4n^3-7}$

e) $u_n = \dfrac{n^2+3n}{n+2} - \dfrac{n^2-1}{n}$

f) $u_n = \dfrac{2n+3}{3n-1}$

g) $u_n = \dfrac{n^2-1}{n^4+3}$

h) $u_n = \dfrac{2^n+1}{2^{n+1}-1}$

i) $u_n = \dfrac{n^3+1}{n^2+2n-1}$

j) $u_n = \dfrac{(-1)^n n^3+1}{n^2+2}$

k) $u_n = \dfrac{n(n-1)(n-2)}{(n+1)(n+2)}$

3.18. Compute, if they exist, the limits of the sequences:

a) $\cos^2(n)\sin\left(\dfrac{1}{n}\right)$,

b) $\dfrac{n(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}$,

c) $(\cos(x))^n$, $x \in \mathbb{R}$,

d) $\dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n+1}} + \ldots + \dfrac{1}{\sqrt{2n}}$,

e) $\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \ldots + \dfrac{1}{\sqrt{n^2+2n+1}}$,

f) $\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \ldots \dfrac{1}{(2n)^2}$,

g) $\dfrac{n}{\sqrt{n^4+1}} + \dfrac{n}{\sqrt{n^4+2}} \ldots + \dfrac{n}{\sqrt{n^4+n}}$,

h) $\sqrt{n+1} - \sqrt{n}$

3.19. Compute, if they exist, the limits of the sequences:

a) $u_n = \left(\dfrac{n+3}{n+1}\right)^{2n}$ b) $u_n = \left(\dfrac{n+5}{2n+1}\right)^n$ c) $u_n = \left(1 - \dfrac{3}{n^2}\right)^n$.

3.20. The sequence $s = \{s_n\}_{n \in \mathbb{N}^*}$ defined by $s_1 = 1$ and $s_n = \sqrt{s_{n-1}+1}$ is convergent. Explain why and compute its limit.

4 Numerical and power series

4.21. Decide if the following series converge. If yes, find the sum.

a) $\sum_{n\geq 0}\left(\frac{1}{2}\right)^n$ b) $\sum_{n\geq 1} 3^n$ c) $\sum_{n\geq 0}\left(\frac{2}{3}\right)^{n+2}$

d) $\sum_{n\geq 3}\left(\frac{1}{4}\right)^n$ e) $\sum_{n\geq 0} 5^n$

4.22. Determine the values of x for which the series are convergent and find the sum.

a) $\sum_{n=0}^{\infty}\left(\frac{x}{x+1}\right)^n$ b) $\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$ c) $\sum_{n=0}^{\infty}(1-|x|^n)$ d) $\sum_{n=0}^{\infty}(x+1)^{2n}$

e) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ f) $\sum_{n=0}^{\infty}\frac{x^n}{n!}+\left(\frac{2}{x}\right)^{n+2}$ g) $\sum_{n=2}^{\infty}\frac{x^n}{n!}$ h) $\sum_{n=0}^{\infty}\frac{(1-|x|)^n}{n!}$

i) $\sum_{n=0}^{\infty}\frac{(x+1)^{2n+6}}{(n+3)!}$ j) $\sum_{n=2}^{\infty}\frac{(x-2)^{n+3}}{n}$

4.23. Use geometrical series to compute the respective rationals:

a) $3,666\ldots$

b) $1,571428571428571428\ldots$

c) $1,181818\ldots$

d) $0,999\ldots$

4.24. Compute the convergence radius and find the largest open set where the following power series are absolutely convergent:

a) $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$ c) $\sum_{n=1}^{\infty}\frac{n(x+1)^{2n}}{3^n}$

b) $\sum_{n=1}^{\infty}\frac{(2x+1)^{2n+1}}{\sqrt{n}}$ d) $\sum_{n=1}^{\infty} n!n^{-n}x^n$