Mathematics I

Exercises

Mathematical Analysis

5 Real functions: domains, limits and continuity

5.1. Determine the domain of the following functions:
a) $\sqrt{x+1}$
b) $\ln(1-x)$
c) $\dfrac{1}{\sqrt{x^2-4}}$
d) $\sqrt{9-x^2}$
e) $\sqrt{e^{x^2}-1}$
f) $\ln(x^2-16)$
g) $\sqrt{|x-2|-4}$
h) $\ln(\ln x)$
i) $\dfrac{1}{\ln(\ln x)-1}$
j) $\dfrac{2}{\sqrt{2-|x-1|}}$

5.2. Compute, if they exist:
a) $\lim\limits_{x\to 0} \sin\dfrac{1}{x}$
b) $\lim\limits_{x\to 0} \left[x\sin\dfrac{1}{x}\right]$
c) $\lim\limits_{x\to +\infty} \left[x\sin\dfrac{1}{x}\right]$
d) $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1}{x}$

5.3. Determine the following limits:
a) $\lim\limits_{x\to \pm\infty} x^2-3x+4$
b) $\lim\limits_{x\to \pm\infty} \dfrac{x^2+x-1}{x-3x^2+4}$
c) $\lim\limits_{x\to \pm\infty} \dfrac{4x^3-5x+1}{2x^2-3x+5}$
d) $\lim\limits_{x\to \pm\infty} \dfrac{x(x^2-1)}{x^2(2x+3)}$
e) $\lim\limits_{x\to -3} \dfrac{x^2+3x}{x^2+6x+9}$

f) $\lim_{x\to 2} \dfrac{x^2-4}{x^3-8}$

m) $\lim_{x\to 2} \dfrac{\sqrt{x+2}-\sqrt{3x-2}}{x-2}$

n) $\lim_{x\to 1} \dfrac{\sqrt{x-1}}{x-1}$

5.4. Consider the functions defined by:

$$h(x) = \begin{cases} 2x + \arccos(x), & 0 \leq x < 1 \\ 2, & x = 1 \\ \dfrac{x+5}{3}, & 1 < x \leq 4 \end{cases}$$

a) Show that h is continuous in all its domain.
b) Using Bolzano theorem, prove that: $\exists c \in]2,4[: h(c) = c$.

5.5. Consider in $\mathbb{R}\setminus\{0\}$ the function $f(x) = \dfrac{1-e^{3x}}{5x}$. What is the value at $x = 0$ so that the extension of f to $\mathbb{R}$ is continuous.

5.6. Consider $f(x) = 1 - x\sin\left(\frac{1}{x}\right)$ on $\mathbb{R}\setminus\{0\}$. Let g be an extension of f to $\mathbb{R}$. Determine $g(0)$ so that g is continuous at $x = 0$.

5.7. Determine a and b that make continuous the following functions at the given points.

a) $f_1(x) = \begin{cases} 3x-7, & x \geq 3 \\ ax+3, & x < 3 \end{cases}, \quad x = 3.$

b) $f_2(x) = \begin{cases} x+a, & x < -2 \\ 3ax+b, & -2 \leq x \leq 1 \\ ax+3, & x > 1 \end{cases}, \quad x = -2,\ x = 1.$

c) $f_3(x) = \begin{cases} \sin(x), & x \leq 0 \\ ax+b, & x > 0 \end{cases}, \quad x = 0$

5.8. Let f be a continuous function from $[a,b]$ to $[a,b]$. Show that there is $c \in [a,b]$ such that $f(c) = c$.

5.9. Prove that any polynomial with an odd degree has at least one root.