Speculative Betas*

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Abstract

We provide a model for why high beta assets are more prone to speculative overpricing than low beta ones. When investors disagree about the common factor of cash-flows, high beta assets are more sensitive to this macro-disagreement and experience a greater divergence-of-opinion about their payoffs. Short-sales constraints for some investors such as retail mutual funds result in high beta assets being over-priced. When aggregate disagreement is low, expected return increases with beta due to risk-sharing. But when it is large, expected return initially increases but then decreases with beta. High beta assets have greater shorting from unconstrained arbitrageurs and more share turnover. Using measures of disagreement about stock earnings and economic uncertainty, we verify these predictions. A calibration exercise yields reasonable parameter values.

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1. Introduction

There is compelling evidence that the risk and return trade-off, the cornerstone of modern asset pricing theory, is often of the wrong sign. This literature, which dates back to Black (1972) and Black et al. (1972), shows that low risk stocks, as measured by a stock's co-movement with the stock market or Sharpe (1964)'s Capital Asset Pricing Model (CAPM) beta, have significantly outperformed high risk stocks over the last thirty years.[1] For instance, Figure 1, analogous to figure 1 c in Baker et al. (2011), shows that the cumulative performance of stocks since January 1968 declines with beta.[2] A dollar invested in a value-weighted portfolio of the lowest quintile of beta stocks would have yielded \$96.21 (\$15.35 in real terms) at the end of December 2010. A dollar invested in the highest quintile of beta stocks would have yielded around \$26.39 (\$4.21 in real terms). This under-performance is as economically significant as famous excess stock return predictability patterns such as the value-growth or price momentum effects.[3]

We provide a theory for this high-risk and low-return puzzle by allowing investors to disagree about the market or common factor of firm cash flows and prohibiting *some* investors from short-selling. First, there is substantial evidence of disagreement among professional forecasters' and households' expectations about many macroeconomic state variables such as market earnings, industrial production growth and inflation (Cukierman and Wachtel (1979), Kandel and Pearson (1995), Mankiw et al. (2004), Lamont (2002)). Macro-disagreement might emanate from many sources such as heterogeneous priors or cognitive biases such as overconfidence.[4] Second, short-sales constraints bind for some investors due to institutional reasons as opposed to the physical cost of shorting.[5] For instance, many investors in the stock market such as retail mutual funds, which in 2010 have 20 trillion dollars under management, are prohibited by charter from shorting directly (Almazan et al. (2004)) or indirectly through the use of derivatives (Koski and Pontiff (1999)). Only a small subset of investors, such as

[1] A non-exhaustive list of studies include Blitz and Vliet (2007), Cohen et al. (2005), and Frazzini and Pedersen (2010).

[2] See section 3.2 for details on the construction of our beta portfolios.

[3] Baker et al. (2011) report that the value-growth effect (Fama and French (1992), Lakonishok et al. (1994)), buying stocks with low price-to-fundamental ratios and shorting those with high ones, generates a reward-to-risk or Sharpe (1964) ratio that is two-thirds of a zero-beta adjusted strategy of buying low beta stocks and shorting high beta stocks. The corresponding figure for the momentum effect (Jegadeesh and Titman (1993)), buying past year winning stocks and shorting past year losing ones, is roughly three-fourths of the long low beta, short high beta strategy.

[4] See Hong and Stein (2007) for a discussion of the various rationales. A large literature starting with Odean (1999) and Barber and Odean (2001) argues that retail investors engage in excessive trading due to overconfidence.

[5] See Lamont (2004) for a discussion of the many rationales for the bias against shorting in financial markets, including historical events such as the Great Depression in which short-sellers were blamed for the Crash of 1929.

hedge funds with 1.8 trillion dollars in asset management, can and do short.

We incorporate these two assumptions — disagreement and short-sales constraint — into an otherwise standard CAPM framework, in which firms' cash flows follow a one-factor model. Investors only disagree about the mean of the macro-factor or common component of cash flows. There are two groups of investors: buyers such as retail mutual funds who cannot short and arbitrageurs such as hedge funds who can short. Our model is the multi-asset extension of Chen et al. (2002)'s rendition of Miller (1977), who originally considered how disagreement and short-sales constraints affects the pricing of a single stock. The key result from these papers is that large divergence of opinion leads to over-pricing because price reflects only the views of the optimists as pessimists are sidelined due to binding short-sales constraints.[6]

Our main result is that high beta assets are over-priced compared to low beta ones when disagreement about the common factor is high. High beta stocks like retailers load more on the macro-factor than low beta companies like utilities. If investors disagree about the mean of the common factor, then their forecasts of the payoffs of high beta stocks will naturally diverge much more than their forecasts of low beta ones. In other words, beta amplifies disagreement about the macro-economy. Because of short-sales constraints, high beta stocks, which are more sensitive to macro-disagreement than low beta ones, are only held in equilibrium by optimists as pessimists are sidelined. This creates over-pricing of high beta stocks compared to low beta ones. Arbitrageurs attempt to correct this mis-pricing but their risk aversion results only in limited shorting leading to equilibrium over-pricing.[7]

Our model yields the following key testable implication. When macro-disagreement is low, all investors are long and short-sales constraints do not bind. The traditional risk-sharing motive leads high beta assets to attract a lower price or higher expected return. For high enough levels of aggregate disagreement, the relationship between risk and return takes on an inverted U-shape. For assets with a beta below a certain cut-off, expected returns are increasing in beta as there is little disagreement about these stock's cash flows and therefore short-selling constraints do not bind in equilibrium. But for assets with a beta above an equilibrium cut-off, disagreement about the dividend becomes sufficiently large that the pessimist investors are sidelined. This speculative over-pricing effect can dominate the risk-sharing effect and the expected returns of high beta assets can actually be lower than

[6]The consideration of a general disagreement structure about both means and covariances of asset returns with short-sales restrictions in a CAPM setting is developed in Jarrow (1980), who shows that short-sales restrictions in one asset might increase the prices of others. It turns out that a focus on a simpler one-factor disagreement structure about common cash-flows yields closed form solutions and a host of testable implications for the cross-section of asset prices that would otherwise not be possible.

[7]High beta stocks might also be more difficult to arbitrage because of incentives for benchmarking and other agency issues (Brennan (1993), Baker et al. (2011).)

those of low beta ones. As disagreement increases, the cut-off level for beta below which all investors are long falls and the fraction of assets experiencing binding short-sales constraints increases.[8]

We test this prediction using monthly time-series of disagreement about market earnings and economic uncertainty. Disagreement for a stock's cash-flow is simply measured by the standard deviation of its analysts' long-term earnings growth forecasts, as in Diether et al. (2002). The aggregate disagreement measure is a beta-weighted average of analyst earnings forecast dispersion for all stocks, similar in spirit to Yu (2010). The weighting by beta in our proxy for aggregate disagreement is suggested by our theory. After all, stocks with very low beta have by definition almost no sensitivity to aggregate disagreement, and their disagreement should mostly reflect *idiosyncratic disagreement*. As can be seen from Figure 5, our time-series of aggregate disagreement is highly correlated with an economic uncertainty series constructed by Bloom (2009) and Bloom et al. (2012), which is simply the cross-sectional standard deviation of U.S. plants sales growth. Note that these measures can be high during both down-markets, like the recessions of 1981-82 and 2007-2008, and up-markets, like the dot-com boom of the late nineties.

As shown in panel (c) of Figure 7, in the months with low aggregate disagreement or uncertainty (defined as the bottom quartile of the disagreement distribution and denoted by blue dots), expected 6-month excess returns are in fact increasing with beta. But in months with high aggregate disagreement or uncertainty (defined as the top quartile of the disagreement distribution and denoted by red dots), the risk-return relationship has an inverted-U shape. For stocks in lowest and highest beta deciles, the average excess return net of the risk-free rate is around 4%. For stocks in middle beta deciles, the average excess return is around 6%. This inverted U-shape relationship is formally estimated in the context of a standard Fama-MacBeth analysis where the concavity of the excess return/β relationship is shown to be strictly increasing with aggregate disagreement.

Our findings are consistent with Diether et al. (2002) and Yu (2010), who find that dispersion of earnings forecasts predicts low returns in the cross-section and for the market return in the time-series respectively, consistent with the predictions of models with disagreement and short-sales constraints. But our focus is on the theory and the empirics of the shape of the Security Market Line as a function of aggregate disagreement, and in particular on its concavity. Importantly, we show below that the inverted U-shape relationship observed in the data is not simply a function of high beta stocks performing badly during down mar-

[8]When aggregate disagreement is so large that pessimists are sidelined on all assets, the relationship between risk and return is entirely downward sloping as the entire market becomes overpriced. We assume that all assets in our model have a strictly positive loading on the aggregate factor. Thus, it is always possible that pessimists want to be short an asset, provided aggregate disagreement is large enough.

kets nor is it a function of high disagreement stocks under-performing. We also show that our finding is not driven by (and if anything made stronger after controlling for) existing cross-sectional predictability patterns like size and value-growth effects in the data. In other words, the inverted-U shaped relationship between beta and return and its dependence on aggregate uncertainty is unique to our model and new to the literature.

Our model also delivers three additional novel predictions, which we also confirm using our aggregate disagreement measures. First, investors' disagreement about the cash flows of high beta assets increases during times of high uncertainty or disagreement about the macro-factor. Second, high beta stocks are more likely to be shorted by arbitrageurs and especially so when aggregate disagreement is high. Third, in an overlapping-generations (OLG) extension of our static model, we show that in high aggregate disagreement states, the share turnover gap between high and low beta assets is higher than in low aggregate disagreement states. Investors anticipate that high beta assets are more likely to experience binding short-sales constraints in the future and hence have a potentially higher resale price than low beta ones relative to fundamentals (Harrison and Kreps (1978), Morris (1996), Scheinkman and Xiong (2003) and Hong et al. (2006)). Since disagreement is persistent, this pushes up the price of high beta assets in the high disagreement state. This overpricing leads arbitrageurs to short high beta assets, thereby increasing the share turnover of these stocks.

We consider a calibration exercise using the OLG extension of our model and show that, under reasonable parameter values, it can generate economically significant concavity in the Security Market Line. Hence, our model provides an alternative to Black (1972)'s model for the high risk and low return puzzle as emanating from leverage constraints. The inverted-U shape between risk and return, predicted by our model and found in the data, is not found in a model based solely on leverage constraints.

Our model also naturally generates market segmentation in the sense that during high uncertainty periods only optimists own high beta stocks. Hence, we deliver an analog to Merton (1987)'s segmented CAPM due to clientele effects, except that volatility attracts lower returns in our setting due to speculation as opposed to higher returns in his setting due to risk absorption. Our insight that high beta assets are more speculative and have higher turnover is related to Hong and Sraer (2011)'s analysis of credit bubbles. They point out how debt, with a bounded upside, is less disagreement sensitive than equity and hence less prone to speculative over-pricing and over-trading.

More generally, our model generates predictions about the pricing of the cross-section of stocks that are different from theories based on risk-sharing, liquidity or even behavioral biases. In Delong et al. (1990), high noise trading risk yields high returns. In Campbell et

al. (1993), high liquidity risk yields high expected return. In Barberis and Huang (2001), mental accounting by investors still leads to a positive relationship between risk and return. The exception is the model of overconfident investors and the cross-section of stock returns in Daniel et al. (2001) that might yield a negative relationship as well but not an inverted U-shape pattern with beta.

Our paper proceeds as follows. We present the model in Section 2. We describe the data in Section 3. We present the empirical analysis in Section 4. We conclude in Section 5. All proofs are in Appendix A.

2. Model

2.1. Static Setting

We consider an economy populated with a continuum of investors of mass 1. There are two periods, $t = 0, 1$. There are N risky assets and the risk-free rate is exogenously set at r. Risky asset i delivers a dividend $\tilde{d}_i$ at date 1, which is given by:

$$\forall i \in \{1, \ldots, N\}, \quad \tilde{d}_i = b_i \tilde{z} + \tilde{\epsilon}_i,$$

where the systematic component is $\tilde{z} \sim \mathcal{N}(\bar{z}, \sigma_z^2)$, the idiosyncratic component is $\tilde{\epsilon}_i \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\text{Cov}(\tilde{z}, \tilde{\epsilon}_i) = 0$. b_i is the cash-flow beta of asset i and is assumed to be strictly positive. Each asset i is in supply $\frac{1}{N}$ and we assume w.l.o.g. that:[9]

$$b_1 < b_2 < \cdots < b_N.$$

Assets in the economy are indexed by their cash-flow betas, which are increasing in i. The value-weighted average b in the economy is set to 1 ($\sum_{i=1}^{N} \frac{b_i}{N} = 1$).

Investors are divided into two groups. A fraction α of them hold heterogenous beliefs and cannot short. We call these buyers mutual funds (MF), who are in practice prohibited from shorting by charter. These investors are divided in two groups of mass $\frac{1}{2}$, A and B, who disagree about the mean value of the aggregate shock $\tilde{z}$. Group A believes that $\mathbb{E}^A[\tilde{z}] = \bar{z} + \lambda$ while group B believes that $\mathbb{E}^B[\tilde{z}] = \bar{z} - \lambda$. We assume w.l.o.g. that $\lambda > 0$ so that group A are the optimists and B the pessimists.

A fraction $1 - \alpha$ of investors hold homogeneous and correct beliefs and are not subject to the short-sales constraint. We index these investors by a (for "arbitrageurs"). For con-

[9] This normalization of supply to $1/N$ is without loss of generality. If asset i is in supply s_i, then what matters is the ranking of assets along the $\frac{b_i}{s_i}$ dimension. The rest of the analysis is then left unchanged

creteness, one might interpret these buyers as hedge funds (HF), who can generally short at little cost.[10] Investors maximize their date-1 wealth and have mean-variance preferences:

$$U(\tilde{W}^k) = \mathbb{E}^k[\tilde{W}^k] - \frac{1}{2\gamma} Var(\tilde{W}^k)$$

where $k \in \{a, A, B\}$ and γ is the investors' risk tolerance. Investors in group A or B maximize under the constraint that their holding of stocks have to be greater than 0.

2.2. Equilibrium

The following theorem characterizes the equilibrium.

Theorem 1. *Let* $\theta = \frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$ *and let* $(u_i)_{i\in[0,N+1]}$ *be a sequence such that* $u_{N+1} = 0$, $u_i = \frac{1}{\gamma N b_i}\left(\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{j<i} b_j^2\right)\right) + \frac{\sigma_z^2}{\gamma}\left(\sum_{j\geq i}\frac{b_j}{N}\right)$ *for* $i \in [1, N]$ *and* $u_0 = \infty$. *u is a strictly decreasing sequence. Let* $\bar{i} = \min\{k \in [0, N+1] \mid \lambda > u_k\}$.

Equilibrium asset prices are given by:

$$P_i(1+r) = \begin{cases} \bar{z}b_i - \frac{1}{\gamma}\left(b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) & \text{for } i < \bar{i} \\ \bar{z}b_i - \frac{1}{\gamma}\left(b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) + \underbrace{\frac{\theta\sigma_\epsilon^2}{\gamma}\left(b_i\left(\frac{\lambda\gamma - \frac{\sigma_z^2}{N}\left(\sum_{i\geq\bar{i}} b_i\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}\right) - \frac{1}{N}\right)}_{\pi^i = \text{speculative premium}} & \text{for } i \geq \bar{i} \end{cases} \tag{1}$$

Assets with high cash-flow betas, i.e. $i \geq \bar{i}$, *are over-priced (relative to the benchmark with no short-sales constraints or when* $\alpha = 0$*) and the amount of over-pricing, defined as the difference between the price and the benchmark price in the absence of short-sales constraints, is increasing with disagreement* λ *and with the fraction of short-sales constrained investors* α.

Group A investors are long all assets. Group B investors are long assets $i < \bar{i} - 1$ *and have 0 holdings of assets* $i \geq \bar{i}$. *There exists* $\hat{\lambda} > 0$ *such that provided that* $\lambda > \hat{\lambda}$, *there exists* $\tilde{i} \in [\bar{i}, N]$ *such that (1) group a investors short high cash-flow beta assets (i.e.* $i \geq \tilde{i}$*) (2) the amount of shorting increases with aggregate disagreement* λ *and (3) the sensitivity of shorting to aggregate disagreement is higher for high cash-flow beta assets.*

Proof. We sketch the derivation of equilibrium prices and leave the remaining proof for the Appendix A.1. We first posit an equilibrium structure and check ex-post that it is indeed an equilibrium. Let $\bar{i} \in [2, N]$. Consider an equilibrium where group B investors are long on assets $i < \bar{i}$ and hold no position (i.e. $\mu_i^B = 0$)

[10] That these investors have homogenous beliefs is simply assumed for expositional convenience. Heterogeneous priors for unconstrained investors wash out in the aggregate and have thus no impact on equilibrium asset prices.

for assets $i \geq \bar{i}$ and group A investors are long all assets $i \in [1, N]$. Since group A investors are long, their holdings satisfy the following first order conditions:

$$\forall i \in [1, N]: \quad (\bar{z} + \lambda) b_i - P_i(1 + r) = \frac{1}{\gamma} \left(\left(\sum_{k=1}^{N} b_k \mu_k^A \right) b_i \sigma_z^2 + \mu_i^A \sigma_\epsilon^2 \right)$$

Since group B investors are long only on assets $i < \bar{i}$, their holdings for these assets must also satisfy the following first order condition:

$$\forall i \in [1, \bar{i} - 1], \quad (\bar{z} - \lambda) b_i - P_i(1 + r) = \frac{1}{\gamma} \left(\left(\sum_{k=1}^{\bar{i}-1} b_k \mu_k^B \right) b_i \sigma_z^2 + \mu_i^B \sigma_\epsilon^2 \right)$$

For assets $i \geq \bar{i}$, group B investors have 0 holdings and so $\mu_i^B = 0$. For these assets, it must be the case that the group B investors' marginal utility of holding the asset, taken at the equilibrium holdings, is strictly negative (otherwise, group B investors would have an incentive to increase their holdings). This is equivalent to:

$$\forall i \geq \bar{i}, \quad (\bar{z} - \lambda) b_i - P_i(1 + r) - \frac{1}{\gamma} \left(\left(\sum_{k=1}^{\bar{i}-1} b_k \mu_k^B \right) b_i \sigma_z^2 \right) < 0$$

Finally, since arbitrageurs are not short-sales constrained, their holdings always satisfy the following first-order condition:

$$\forall i \in [1, N]: \quad \bar{z} b_i - P_i(1 + r) = \frac{1}{\gamma} \left(\left(\sum_{k=1}^{N} b_k \mu_k^a \right) b_i \sigma_z^2 + \mu_i^a \sigma_\epsilon^2 \right)$$

The market clearing condition for asset i is simply: $\alpha \frac{\mu_i^A + \mu_i^B}{2} + (1 - \alpha) \mu_i^a = \frac{1}{N}$. We sum the first-order conditions of investors a, A and B for assets $i < \bar{i}$, and of investors a and A only for assets $i \geq \bar{i}$, weighting the sum by the size of each investors group (i.e. $\frac{\alpha}{2}$ for group A and B and $1 - \alpha$ for group a). This results in the following equations:

$$\begin{cases} \bar{z} b_i - P_i(1 + r) = \frac{1}{\gamma} \left(b_i \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} \right) & \text{for } i < \bar{i} \\ \left(1 - \frac{\alpha}{2} \right) (\bar{z} b_i - P_i(1 + r)) + \frac{\alpha}{2} \lambda b_i = \frac{1}{\gamma} \left(b_i \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} - \frac{\alpha}{2} \sigma_z^2 b_i \sum_{k=1}^{\bar{i}-1} b_k \mu_k^B \right) & \text{for } i \geq \bar{i} \end{cases} \tag{2}$$

Call $S = \sum_{i=1}^{\bar{i}-1} b_i \mu_i^B$. S represents the exposure of group B investors to the aggregate factor $\tilde{z}$. We look for an expression for S. We start by using the first order conditions of group B investors on assets $i < \bar{i}$ and plug in the equilibrium price of assets $i < \bar{i}$ found in the first equation of system (2):

$$\forall i < \bar{i}, \qquad -\lambda \gamma b_i + b_i \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} = S b_i \sigma_z^2 + \mu_i^B \sigma_\epsilon^2$$

We can now simply multiply the previous equation by b_i for all $i < \bar{i}$ and sum up the resulting equations for $i < \bar{i}$, which results in:

$$S \sigma_\epsilon^2 = -\lambda \gamma \left(\sum_{k < \bar{i}} b_k^2 \right) - S \sigma_z^2 \left(\sum_{k < \bar{i}} b_k^2 \right) + \sigma_z^2 \left(\sum_{k < \bar{i}} b_k^2 \right) + \frac{\sigma_\epsilon^2}{N} \left(\sum_{k \geq \bar{i}} b_k \right) \tag{3}$$

From the previous expression, we can now derive S:

$$S = 1 - \frac{\sigma_\epsilon^2 \left(\sum_{i \geq \bar{i}} \frac{b_i}{N}\right) + \lambda\gamma \left(\sum_{i<\bar{i}} b_i^2\right)}{\sigma_\epsilon^2 + \sigma_z^2 \left(\sum_{i<\bar{i}} b_i^2\right)}$$

Now that we have a closed-form expression for S, we simply plug it into the second equation of system 2 for assets $i \geq \bar{i}$, allowing us to obtain the equilibrium price of assets $i \geq \bar{i}$:

$$P_i(1+r) = \bar{z}b_i - \frac{1}{\gamma}\left(b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) + \underbrace{\theta\left(b_i \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}\left(\lambda - \frac{\sigma_z^2}{\gamma N}\sum_{k \geq \bar{i}} b_k\right) - \frac{\sigma_\epsilon^2}{\gamma N}\right)}_{\pi^i = \text{speculative premium}} \tag{4}$$

where $\theta = \frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$ is a strictly increasing function of α. The first equation of system 2 provides us with a simple expression for the price of assets $i < \bar{i}$:

$$P_i(1+r) = \bar{z}b_i - \frac{1}{\gamma}\left(b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) \tag{5}$$

The proposed equilibrium holds provided that group B investors have (1) strictly positive holdings of asset $\bar{i}-1$ and (2) negative marginal utility of holding asset $\bar{i}$ at the equilibrium holding. We show in Appendix A.1 that this is equivalent to $\bar{i} = \min\{k \in [0, N+1] | \lambda > u_k\}$ where $(u_k)_{k\in[0,N+1]}$ is defined in Theorem 1. The rest of the proof is provided in Appendix A.1. □

The dividends of high b_i assets are more sensitive to aggregate disagreement (λ). Thus there is more disagreement among investors about the expected dividends of high b_i assets relative to low b_i assets. Above a certain level of b_i ($b_i \geq b_{\bar{i}}$), investors sufficiently disagree that group B would like to optimally short these stocks. However, this is impossible because of the short-sales constraint. These stocks thus experience a speculative premium since their price reflects only the belief of the optimistic group A. As aggregate disagreement grows, the cash flow beta of the marginal asset — the asset above which group B investors are sidelined — decreases, i.e. there is a larger fraction of assets experiencing over-pricing.

When short-sales constraints are binding, i.e. for these assets $i \geq \bar{i}$, the difference between the unconstrained price and the constrained price is given by:

$$\pi^i = \frac{\theta\sigma_\epsilon^2}{\gamma}\left(b_i\left(\frac{\lambda\gamma - \frac{\sigma_z^2}{N}\left(\sum_{i\geq\bar{i}} b_i\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}\right) - \frac{1}{N}\right).$$

This term, which we call the speculative premium, captures the degree of over-pricing due to the short-sales constraints. This speculative premium is strictly increasing with θ, i.e. with the fraction of short-sales constrained investors α. For a fixed b_i, the larger is the divergence of opinion λ, the greater the over-pricing.

The final result in Theorem 1 looks at the amount of shorting in equilibrium and how it

is impacted by aggregate disagreement. Intuitively, HFs (i.e. group a investors) short assets with large mispricing, i.e. high b assets. As aggregate disagreement increases, mispricing increases, so that HFs end up shorting more. However, an increase in aggregate disagreement leads to a larger increase in mispricing of high b stocks, so that the corresponding increase in shorting is larger for high b stocks as well. In other words, there is in general a weakly increasing relationship between shorting by HFs and b. Provided that λ is large enough, this relationship becomes strictly steeper as aggregate disagreement increases.

2.3. Risk and Expected Return

We now restate equilibrium prices in terms of expected excess returns and relate them to the familiar market β from the CAPM. Define excess returns as $\tilde{R}_i = b_i\tilde{z} - (1+r)P_i$. Define β_i such that $\beta_i = \frac{\text{Cov}(\tilde{R}_i, \tilde{R_M})}{Var(\tilde{R_M})}$ and $\tilde{R_M} = \sum_{i=1}^{N} \frac{\tilde{R}^i}{N}$. Finally, define $\kappa(\lambda) = \frac{\sigma_\epsilon^2}{\sigma_z^2} \frac{\lambda\gamma - \frac{\sigma_z^2}{N}\left(\sum_{i\geq\bar{i}} b_i\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)} > 0$. Then expected excess returns are given by:[11]

$$\mathbb{E}[\tilde{R}_i] = \begin{cases} \beta_i \dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma} & \text{for } i < \bar{i} \\ \beta_i \dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma}\left(1 - \theta\kappa(\lambda)\right) + \dfrac{\sigma_\epsilon^2}{\gamma N}\theta\left(1 + \kappa(\lambda)\right) & \text{for } i \geq \bar{i} \end{cases} \tag{6}$$

This follows directly from Theorem 1. For $\alpha = 0$ ($\theta = 0$), investors have homogenous beliefs so that λ does not affect the expected returns of the assets and the standard CAPM formula holds. Precisely, the expected returns depend on the covariance of the asset with the market return, and the risk premium is simply determined by the ratio of the variance of the market return (which is close to the variance of the aggregate factor σ_z^2 when N is large) to the risk tolerance of investors γ.

However, when a fraction $\alpha > 0$ of investors are short-sales constrained and aggregate disagreement is large enough, $\bar{i} \leq N$ and expected returns for assets $i \geq \bar{i}$ depend on aggregate disagreement λ. High beta stocks are subject to more disagreement about their expected cash flows and experience more binding short-sales constraints, higher prices and hence lower expected returns. The CAPM does not hold and the Security Market Line is kink-shaped. For assets with a beta above some cut-off ($\bar{i}$ is determined endogenously and depends itself on λ), the expected return is increasing with beta but at a lower pace than for assets with a beta below this cut-off (this is the $-\theta\kappa(\lambda) < 0$ term above). If λ is large enough, the slope of the Security Market Line for assets $i \geq \bar{i}$ can even be negative, i.e. the Security Market Line is inverted-U shaped.

[11] The derivation of this formula can be found in Appendix A.2.

In our empirical analysis below, we approach this kink-shaped relationship between expected excess returns and β by looking at the concavity of the Security Market Line and how this concavity is related to our proxies for aggregate disagreement. In addition, rather than relying fully on the structure of the model, we take a simpler approach and also estimate directly the slope of the Security Market Line, i.e. the coefficient estimate of an OLS regression of realized excess returns on β. We show in the following corollary that that our model predicts that this coefficient is strictly decreasing with idiosyncratic disagreement.

Corollary 1. *Let $\hat{\mu}$ be the coefficient estimate of a cross-sectional regression of realized returns $\tilde{R}_i$ on β_i (and assuming there is a constant term in the regression). The coefficient $\hat{\mu}$ decreases with λ the aggregate disagreement. This effect is larger when there are less arbitrageurs in the economy (i.e. when α increases).*

Proof. See Appendix A.3 □

In the absence of disagreement and short-sales constraints ($\alpha = 0$ – i.e. all investors are arbitrageurs), the slope of the Security Market Line is simply $\hat{\mu} = \frac{\sigma_z^2 + \left(\sum_{i=1}^{N} \frac{1}{N^2}\right)\sigma_\epsilon^2}{\gamma}$. When at least one asset experiences binding short-sales constraints (i.e. when $\lambda \geq u_N$), $\hat{\mu}$, the slope of the Security Market Line (as estimated from a regression of excess returns on β), is strictly decreasing with λ, the aggregate disagreement parameter. In particular, it is direct to show that $\hat{\mu}$ will be strictly negative, provided that λ is large enough relative to γ (i.e. that the speculation motive for trading is large relative to the risk-sharing motive for trading). Furthermore, the role of aggregate disagreement is magnified by the presence of short-sales constrained investors: in an economy with a low fraction of arbitrageurs (i.e. high short-sales constraints), an increase in λ leads to a much larger decrease in the estimated slope of the Security Market Line than in an economy with many arbitrageurs.

Corollary 1 implies that: (1) the slope of the SML ($\hat{\mu}$) is strictly lower when short-sales constraints are binding ($\lambda > u_N$) than in the absence of binding short-sales constraints ($\lambda < u_N$) and (2) the slope of the Security Market Line is strictly decreasing with λ as soon as $\lambda > u_N$. In particular, provided λ is high enough, the estimated slope of the Security Market Line $\hat{\mu}$ can even become negative.

2.4. Discussion of Assumptions and Limiting Cases

Our theory relies on two fundamental ingredients, disagreement and short-sales constraint. Both are important. In the absence of disagreement, all investors share the same portfolio and since there is a strictly positive supply of assets, this portfolio is long only. Thus, the short-sales constraint is irrelevant – it never binds – and the standard CAPM results apply.

In the absence of short-sales constraints, the disagreement of group A and group B investors washes out in the market clearing condition and prices simply reflect the average belief, which we have assumed to be correct.

The model also relies on important simplifying assumptions. First of them is that, in our framework, investors disagree only on the expectation of the aggregate factor, $\tilde{z}$. A more general setting would allow investors to also disagree on the idiosyncratic component of stocks dividend $\tilde{\epsilon}_i$. If the idiosyncratic disagreement on a stock is not systematically related to this stock's cash-flow beta, then our analysis is left unchanged since whatever mispricing is created by idiosyncratic disagreement, it does not affect the shape of the Security Market Line in a systematic fashion. If idiosyncratic disagreement is positively correlated with stocks' cash-flow beta, then the impact of aggregate disagreement on the Security Market Line becomes even stronger. This is because there are now two sources of over-pricing linked systematically with b_i: one coming from aggregate disagreement, the other coming from this additional idiosyncratic disagreement. Of course, theoretically, the case where idiosyncratic disagreement is negatively correlated with b_i – low cash flow beta assets have high idiosyncratic disagreement – can potentially reverse our result. In this case, high beta assets are overpriced because of their higher exposure to aggregate disagreement, but low beta assets could also be overpriced because of their higher exposure to idiosyncratic disagreement. However, as we show in Figure 6(b), in the data, disagreement on high β stocks earnings is always larger than on low beta stocks, *even* in low aggregate disagreement months. This suggests that, if anything, idiosyncratic disagreement is larger for high beta stocks. We also believe that this conforms to standard intuition on the characteristics of high and low beta stocks.

Another restriction in the model is that investors only disagree on the first moment of the aggregate factor $\tilde{z}$ and not on the second moment σ_z^2. From a theoretical viewpoint, this is not very different. In the same way that β scales disagreement regarding $\bar{z}$, β would scale disagreement about σ_z^2. In other words, label the group that underestimates σ_z^2 as the optimists and the group that overestimate σ_z^2 as the pessimists. Optimists are more optimistic about the utility derived from holding a high β asset than a low β asset and symmetrically, the pessimists are more pessimistic about the utility derived from holding a high β asset than a low β asset. Again, high β assets are more sensitive to disagreement about the variance of the aggregate factor σ_z^2 than low β assets. As in our model, this would naturally lead to high β stocks being overpriced when this disagreement about σ_z^2 is large. However, while empirical proxies for disagreement about the mean of the aggregate factor can be constructed, it is not clear how one would proxy for disagreement about its variance.

The model also imposes homoskedasticity of the dividend process. If dividends are

heteroskedastic, then the short-sales constraints need not bind on high beta assets first. More precisely, it is easily shown that in equilibrium short-sales constraints bind on assets with a large ratio of cash-flow beta b_i to idiosyncratic volatility of stock i's dividend process, $\sigma^2_{\epsilon,i}$. That is, one can simply re-rank the assets in ascending order of $b_i/\sigma^2_{\epsilon,i}$, $\frac{b_N}{\sigma^2_{\epsilon,N}} > \frac{b_{N-1}}{\sigma^2_{\epsilon,N-1}} > \cdots > \frac{b_1}{\sigma^2_{\epsilon,1}}$ and the equilibrium has a marginal asset $\bar{i}$ such that assets $i \geq \bar{i}$ experience binding short-sales constraints. However, the data suggest a monotonic relationship between betas and idiosyncratic volatility. Figure 2 shows that the median stock-level $\frac{\beta}{\sigma^2_{\epsilon,i}}$ increases with the average β of our 20 β portfolios defined over the 1981 to 2010 period.[12] We thus maintain our homoskedasticity assumption throughout the paper though our model easily accommodates heteroskedasticity in the dividend process.[13]

Finally, it is interesting to consider the limiting case where σ^2_ϵ goes to 0 to understand the role idiosyncratic risk plays in our model. In that case, the price of asset i (which delivers $b_i\tilde{z}$) has to be equal to b_j/b_i times the price of asset j (which delivers $b_j\tilde{z}$).[14] Thus, the Security Market Line is necessarily *linear* when $\sigma^2_\epsilon = 0$, since for all $i > 1$: $P_i = b_i \times \frac{P_1}{b_1}$. However, it can still be that *all* assets are over-priced. If this is the case, then group B investors are out of the stock market. This is possible only if, for all $i \in [1, N]$, their marginal utility of holding stock i at 0 is strictly negative, i.e.: $b_i(\bar{z} - \lambda) - (1 + r)P_i < 0$. Using the constrained price derived in equation 4 with $\sigma^2_\epsilon = 0$, we see that in that case: $P_i(1 + r) = b_i\bar{z} - \frac{b_i}{\gamma}\left(\sigma^2_z - \theta\left(\lambda\gamma - \sigma^2_z\right)\right)$. The condition for stock i to be over-priced then becomes: $-\lambda + \frac{1}{\gamma}\left(\sigma^2_z - \theta\left(\lambda\gamma - \sigma^2_z\right)\right) < 0 \Leftrightarrow \lambda > \frac{\sigma^2_z}{\gamma} = u_1$. Thus, when $\sigma^2_\epsilon = 0$, two cases arise. When aggregate disagreement is larger than the aggregate variance $\frac{\sigma^2_z}{\gamma}$, all assets are over-priced. The Security Market Line is a line with slope $\frac{1}{\gamma}\left(\sigma^2_z - (1 + \theta)\left(\lambda\gamma - \sigma^2_z\right)\right)$. Group B investors are sidelined from the stock market. There is a CAPM representation of stock returns, but the market return is smaller than in the standard CAPM and can even be strictly negative. The second case is when $\lambda\gamma < \sigma^2_z$ and the standard CAPM results apply. While over-pricing survives in this limiting case, the kinked-shape relationship between expected excess return and β no longer exists.

[12] We defer the reader to Section 3.2 for the construction of our 20 β portfolios.

[13] In particular, closed forms are also obtained under heteroskedasticity. The general model with heteroskedastic dividends is available from the authors upon request.

[14] Either $\theta < 1$ and then HFs can arbitrage away any deviation from $b_iP_j = b_jP_i$ by constructing a long short portfolio delivering 0 for sure at date 1 and yielding them a strictly positive profit at date 0. Or $\theta = 1$, i.e. all investors are long only, and then any deviation from $b_iP_j = b_jP_i$ results in 0 demand for the asset with the larger price per unit of exposure to $\tilde{z}$, which makes the deviation unsustainable.

2.5. Dynamic Setting

2.5.1. Set-up

Consider now a dynamic extension of the previous model. We consider an overlapping generation framework. Time is infinite, $t = 0, 1, \ldots \infty$. Each period t, a new generation of mass 1 is born and invest in the stock market to consume the proceeds at date $t + 1$. Thus at date t, the new generation is buying assets from the current old generation (born at date $t - 1$), which has to sell its entire portfolio in order to consume. Each generation is composed with 2 groups of investors: arbitrageurs, or Hedge Funds, in proportion $1-\alpha$, and Mutual funds in proportion α. Investors have mean-variance preferences with risk tolerance parameter γ. There are N assets, whose dividend process is given by:

$$\tilde{d}^i = b^i \tilde{z} + \tilde{\epsilon}^i$$

The timeline of the model appears on Figure 3. Mutual funds born at date t hold heterogeneous beliefs about the expected value of $\tilde{z}_{t+1}$. Specifically, there are two groups of mutual funds: group A – the optimist MFs – whose expectation about $\tilde{z}_{t+1}$ is given by $\mathbb{E}^A[\tilde{z}_{t+1}] = \bar{z} + \tilde{\lambda}_t$) and group B – the pessimist MFs – whose expectation about $\tilde{z}_{t+1}$ is given by $\mathbb{E}^B[\tilde{z}_{t+1}] = \bar{z} - \tilde{\lambda}_t$).[15] Finally, we assume that $\tilde{\lambda}_t \in \{0, \lambda > 0\}$ is a two-state Markov process with persistence $\rho \in]1/2, 1[$.

2.5.2. Risk and Share Turnover

Call $P_t^i(\tilde{\lambda})$ the price of asset i at date t when realized aggregate disagreement is $\tilde{\lambda}_t = \tilde{\lambda}$ and $\Delta P_t^i = P_t^i(\lambda) - P_t^i(0)$. Call $\mu_i^k(\tilde{\lambda}_t)$ the number of shares of asset i purchased by investors in group k when realized aggregate disagreement is $\tilde{\lambda}_t$. Call λ_t^k the realized belief at date t for group k.[16] We first compute the date-t+1 consumption of investors in group $k \in \{a, A, B\}$ born at date t:

$$\tilde{W^k} = \left(\sum_{i \leq N} \mu_i^k(\tilde{\lambda}_t) b_i \right) \tilde{z} + \sum_{i \leq N} \mu_i^k(\tilde{\lambda}_t) \tilde{\epsilon}^i + \sum_{i \leq N} \mu_i^k(\tilde{\lambda}_t) \left(P_{t+1}^i(\tilde{\lambda}_{t+1}) - (1+r) P_t^i(\lambda_t) \right)$$

[15] Most of the assumptions made in this model are discussed in Section 2.1 in the context of our static model.

[16] $\lambda_t^A = \lambda$ and $\lambda_t^B = -\lambda$ when $\tilde{\lambda}_t = \lambda$ or $\lambda_t^A = \lambda_t^B = 0$ when $\tilde{\lambda}_t = 0$ and $\lambda_t^a = 0$

Thus, the expected consumption at date $t+1$ from investor k's viewpoint, and its associated variance are given by:

$$\begin{cases} \mathbb{E}^k[\tilde{W}^k] = \left(\sum_{i\leq N} \mu_i^k(\tilde{\lambda}_t) b_i\right)(\bar{z} + \lambda_t^k) + \sum_{i\leq N} \mu_i^k(\tilde{\lambda}_t)\left(\mathbb{E}[P_{t+1}^i(\tilde{\lambda}_{t+1})|\tilde{\lambda}_t] - (1+r)P_t^i(\lambda_t)\right) \\ Var[\tilde{W}^k] = \left(\sum_{i\leq N} \mu_i^k(\tilde{\lambda}_t) b_i\right)^2 \sigma_z^2 + \sum_{i\leq N} (\mu_i^k(\tilde{\lambda}_t))^2 \sigma_\epsilon + \rho(1-\rho)\left(\sum_{i\leq N} \mu_i^k(\tilde{\lambda}_t)\left(\Delta P_{t+1}^i\right)\right)^2 \end{cases}$$

Relative to the static model, there are two notable changes. First, investors value the resale price of their holding at date 1 (the $\mathbb{E}[P_{t+1}^i(\tilde{\lambda}_{t+1})|\tilde{\lambda}_t]$ term in expected consumption). Second, investors now bear the corresponding risk that the resale prices move with aggregate disagreement $\tilde{\lambda}_t$ (this is in our binomial setting the $\rho(1-\rho)\left(\sum_{i\leq N} \mu_i^k(\tilde{\lambda}_t)\left(\Delta P_{t+1}^i\right)\right)^2$ term in consumption variance).

The following Theorem characterizes the equilibrium of this economy:

Theorem 2. *Let $\bar{i} = \min\{k \in [0, N+1] \mid \lambda > u_k\}$. Short-sales constraints bind only for the group of pessimist investors (i.e. group B), in the high disagreement states ($\tilde{\lambda}_t = \lambda > 0$) and for assets $i \geq \bar{i}$. Denote by π^j the speculative premium, $\pi^j = \theta\left(b_j \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}\left(\lambda - \frac{\sigma_z^2}{\gamma N}\sum_{k\geq\bar{i}} b_k\right) - \frac{\sigma_\epsilon^2}{\gamma N}\right)$ and define:*

$$\Gamma^\star = \frac{-(1+r) + (2\rho-1) + \sqrt{\left((1+r) - (2\rho-1)\right)^2 + \frac{4}{N}\frac{\theta\rho(1-\rho)}{\gamma}\sum_{j\geq\bar{i}}\pi^j}}{2\frac{\theta\rho(1-\rho)}{\gamma}}$$

Equilibrium returns are given by:

$$\begin{cases} \mathbb{E}[R^j(\lambda)] = \mathbb{E}[R^j(0)] = \frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) & \text{for } j < \bar{i} \\ \mathbb{E}[R^j(0)] = \frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j\right) & \text{for } j \geq \bar{i} \\ \mathbb{E}[R^j(\lambda)] = \frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j\right) - \frac{1+r-(2\rho-1)}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j & \text{for } j \geq \bar{i} \end{cases}$$

Proof. See Appendix A.4. □

Low b assets (i.e. $j < \bar{i}$) are never shorted since there is not enough disagreement among investors to make the pessimist investors willing to go short, even in the high disagreement states. Thus, the price of these assets is the same in both states of nature and similar to the standard CAPM case. High b assets (i.e. $j \geq \bar{i}$) experience binding short-sales constraints by

group B investors in the high disagreement state ($\tilde{\lambda} = \lambda > 0$).[17]. This leads to overpricing of these assets relative to the standard benchmark without disagreement.

A consequence of the previous analysis is that the price of assets $j \geq \bar{i}$ depends on the realization of aggregate disagreement. There is thus an extra source of risk embedded in these assets: their resale price is more exposed to aggregate disagreement. These assets are riskier and command an extra risk premium relative to lower b assets. This extra risk premium is given by: $\frac{1}{\gamma}\rho(1-\rho)\frac{\Gamma^\star}{(1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j$. Relative to a benchmark without disagreement (and where expected returns are always equal to $\frac{1}{\gamma}\left(b_j\sigma_z^2+\frac{\sigma_\epsilon^2}{N}\right)$), high b assets have higher returns in low disagreement states (because of the extra risk-premium). In high disagreement states, the returns of high b assets are strictly lower than in low disagreement states, since the large disagreement about next-period dividends lead to overpricing. Thus, the slope of the Security Market Line strictly decreases for high b assets in high disagreement states, while it remains unchanged for low b assets. Whether the returns of high b assets are lower or higher than in the benchmark without disagreement depends on the relative size of the extra-risk premium and the speculative premium. In the data, however, aggregate disagreement is persistent, i.e. ρ is close to 1. A first-order taylor expansion of $\Gamma^\star$ gives that $\Gamma^\star \approx \sum_{j\geq\bar{i}}\frac{\pi_j}{N}$ so that provided that ρ is close to 1, $\mathbb{E}[R^j(\lambda)] < \frac{1}{\gamma}\left(b_j\sigma_z^2+\frac{\sigma_\epsilon^2}{N}\right)$. Intuitively, when aggregate disagreement is persistent, this resale price risk is very limited, since there is only a small probability that the price of high b assets will change next period. Thus, the speculative premium term dominates and returns of high b assets are lower than under the no-disagreement benchmark. We summarize these findings in the following proposition:

Proposition 1. *Let $\bar{i} = \min\{k \in [0, N+1] \mid \lambda > u_k\}$. Then the following holds: (i) The Security Market Line is strictly increasing in low disagreement states ($\tilde{\lambda} = 0$). The slope of the Security Market Line in these states is strictly higher for assets $j \geq \bar{i}$ than for assets $j < \bar{i}$. (ii) The Security Market Line is strictly increasing in high disagreement states ($\tilde{\lambda} = \lambda > 0$) for assets $j < \bar{i}$. For assets $j \geq \bar{i}$, the slope of the Security Market Line in high disagreement states can be either higher or lower than for assets $j < \bar{i}$. There exists $\rho^\star < 1$ such that for $\rho \geq \rho^\star$, the slope of the Security Market Line is strictly lower for high β assets than for low β assets. (iii) The Security Market Line can slope down for assets $j \geq \bar{i}$ in high disagreement states provided ρ is close to 1 and λ is large enough. (iv) The slope of the Security Market Line for assets $j \geq \bar{i}$ is strictly lower in high disagreement states than in low disagreement states.*

Proof. See Appendix A.5. □

[17] In the low disagreement state, $\tilde{\lambda} = 0$ so there is no disagreement among investors and hence there cannot be any binding short-sales constraint.

We next turn to our final result relating to turnover. Turnover in this economy is defined as the number of shares exchanged in each period between the old and the new generation. If all investors are long only, then this is trivially given by the supply of shares, i.e. $\frac{1}{N}$. Indeed, the old generation is supplying $\frac{1}{N}$ shares of each asset and investors in the new generation are simply buying all these shares. Thus, in the low disagreement state, turnover is always equal to $\frac{1}{N}$. When HFs short some assets in the high disagreement state, the effective number of shares for these assets that are exchanged on the market become strictly larger than $\frac{1}{N}$. In particular, turnover in this case is simply given by: $\frac{1}{N} + |\mu^a(\lambda)|$, provided that $\mu^a(\lambda) < 0$. In this context, the following proposition characterizing turnover can be easily shown:

Proposition 2. *Let $\bar{i} = \min\{k \in [0, N+1] \mid \lambda > u_k\}$. Turnover is constant equal to 1/N in the low disagreement state. In the high disagreement state, there exists $\hat{i} \geq \bar{i}$ such that turnover is strictly greater than 1/N and strictly increasing with b for assets $j \geq \hat{i}$. In other words, the differential turnover between high b assets ($j \geq \hat{i}$) and low b assets ($j < \hat{i}$) is strictly greater in the high disagreement state than in the low disagreement state.*

These results are the exact mirrors of the results on shorting by group a investors in Theorem 1. They are a direct consequence of the proof of Theorem 2 and are left to the reader.

2.6. Calibration

In this section, we present a simple calibration of the dynamic version of the model. The objective of this calibration is to see what magnitude of aggregate disagreement is required to obtain a significant distortion in the Security Market Line.

The parameter we use are the following. ρ is set to .94. This corresponds to the coefficient of an AR(1) regression of the proxy for aggregate disagreement we use in our empirical analysis. We set the number of assets to $N = 100$, which is arbitrary. σ_z^2 is set to .0022 and σ_ϵ^2 to .029. These two values correspond to the empirical aggregate and idiosyncratic variance of monthly stock returns over the 1970-2010 period.[18] $\alpha \approx .63$ (i.e. $\theta = 1.75$) corresponds to the fraction of the stock market held by mutual funds and retail investors, for which the cost of shorting is presumably non-trivial. Finally, γ, the risk-tolerance parameter is set to match the average in-sample monthly stock excess returns observed in the data of .5% monthly.

Three cases are displayed on Figure 4:

1. $\lambda = 0.05$. γ is then set to .48 to match the equity premium over 1970-2011. There are 56 assets shorted at equilibrium. This level of disagreement corresponds roughly

[18] Aggregate variance is the variance of the market return. Idiosyncratic variance is the variance of the residual of a CAPM regression of monthly returns on the contemporaneous market return.

to 10% of the standard deviation of the market return. We see on Figure 4(a) that for this level of disagreement, the distortion on the Security Market Line is limited.

2. $\lambda = 0.10$. γ is then set to .347 to match the equity premium over 1970-2011. There are 88 assets shorted at equilibrium. This level of disagreement corresponds to 21% of the standard deviation of the market return. We see on Figure 4(b) that for this level of disagreement, the distortion on the Security Market Line becomes noticeable. However, the Security Market Line at this level of disagreement is still upward slopping for all β.

3. $\lambda = 0.15$. γ is then set to .29 to match the equity premium over 1970-2011. There are 93 assets shorted at equilibrium. This level of disagreement corresponds to 32% of the standard deviation of the market return. We see on Figure 4(c) that for this level of disagreement, the Security Market Line has an inverted-U shape.

Two additional remarks can be made on these calibrations. First, while the Security Market Line in our theory is not piecewise linear but rather piecewise quadratic [19], it appears in our calibration as a piecewise linear function. This is because the quadratic coefficient is divided by N and with $N = 100$, it is not noticeable. Second, the extra-risk premium coming from the exposure of high β assets to changes in aggregate disagreement is negligible at the level of persistence ρ that we observe in the data (i.e. $\rho = .94$).

2.7. Empirical Predictions

In this section, we simply summarize the empirical predictions of our model, which we will take to the data. The first prediction – which is almost an assumption in our model – is that β scales up aggregate disagreement:

Prediction 1. *High beta stocks experience more disagreement in months with high aggregate disagreement.*

The next two predictions, taken from the dynamic model, tie together shorting, turnover, β and aggregate disagreement:

Prediction 2. *There is an increasing relationship between shorting and β. This relationship is steeper in high disagreement months.*

Prediction 3. *There is an increasing relationship between turnover and β. This relationship is steeper in high disagreement months.*

[19] The quadratic term corresponds to the $\frac{b_i}{N} \frac{\sum_{i>\bar{i}} b_i}{\sigma_\epsilon^2 + \sigma_z^2 \sum_{i<\bar{i}} b_i^2}$ term in the speculative premium

The fourth prediction describes the Security Market Line as a function of aggregate disagreement. It holds exactly true in the static model and holds true in the dynamic model provided ρ is large enough. In particular, it holds true in our calibration of the model, which uses the empirical estimation of ρ.

Prediction 4. *In low disagreement months, the Security Market Line is upward slopping. In high disagreement months, the Security Market Line has a kink-shape: its slope is strictly positive for low β assets, but strictly lower (and potentially negative) for high β assets.*

The final prediction relates the average slope of the Security Market Line with aggregate disagreement. It holds true in both the static and the dynamic model.

Prediction 5. *The average slope of the Security Market Line is strictly higher in low disagreement months than in high disagreement months.*

3. Data

In this section, we present the data used in this paper. Table 1 presents descriptive statistics of the variables used in subsequent analyses.

3.1. Data Source

The data in this paper are collected from two main sources. U.S. stock return data are from the CRSP tape and include all available common stocks on CRSP between January 1970 and December 20010. We exclude penny stocks defined as stocks with a share price below \$5. β's are computed with respect to the value-weighted market returns provided on Ken French's website. Excess returns are above the US Treasury bill rate, which we also download from his website. Monthly turnover is defined as monthly volume normalized by number of shares outstanding.[20] We also use stock analyst forecasts of the earnings-per-share (EPS) long-term growth rate (LTG) as the main proxy for investors beliefs regarding the future prospects of individual stocks. The data are provided in the I/B/E/S database. As explained in detail in Yu (2010), the long-term forecast has several advantages. First, it features prominently in valuation models. Second, it is less affected by a firms earnings guidance relative to short-term forecasts. Because the long-term forecast is an expected growth rate, it is directly comparable across firms or across time. We use analyst forecasts from December 1981 through December 2010. Finally, we obtain short-interest ratio data from Bloomberg. These data cover 17,716 stocks over the 1988-2009 period.

[20]For NASDAQ stocks, we take 1/2 the volume to share outstanding in calcuating turnover as is the convention when using CRSP data.

3.2. Constructing β portfolios

We follow the literature in constructing beta portfolios in the following manner. Each month, we use the past twelve months of daily returns to estimate the market beta of each stock in that cross-section. This is done by regressing, at the stock-level, the stock's excess return on the contemporaneous excess market return as well as five lags of the market return to account for the illiquidity of small stocks (Dimson (1979)). Our measure of β is then the sum of these six OLS coefficients. To limit the influence of large absolute returns on the estimation of these βs, we windsorize the daily returns using as thresholds the median return +/- five times the interquartile range of the daily return distribution.[21]

We then sort stocks into 20 beta portfolios based on these pre-ranking betas. We compute the daily equal-weighted returns on these portfolios. We then compute the post-ranking βs by regressing each portfolio daily returns on the excess market returns, as well as five lags of the market return and adding these six estimates. These post-ranking βs are computed using the entire sample period (Fama and French (1992)). Using a one-year trailing window to compute post-ranking time-varying β's give quantitatively similar but noisier estimates.

3.3. Measuring Aggregate Disagreement and Uncertainty

Our measure of aggregate disagreement is similar in spirit to Yu (2010). Stock-level disagreement is measured as the dispersion in analyst forecasts of the earnings-per-share (EPS) long-term growth rate (LTG). We then aggregate this stock-level disagreement measure, weighting each stock by its post-ranking β. Intuitively, our model suggests that there are two components to the overall disagreement on a stock dividend process: (1) a first component coming from the disagreement about the aggregate factor $\tilde{z}$ – the λ in our model and (2) a second component coming from disagreement about the idiosyncratic factor $\tilde{\epsilon^i}$. We are interested in constructing an empirical proxy for the first component only (see our discussion in Section 2.4). To that end, disagreement about low β stocks should only play a minor role since disagreement about a low β stock has to come mostly from idiosyncratic disagreement (in the limit, disagreement about a $\beta = 0$ stock can only come from idiosyncratic disagreement). Thus, we weight each stock-level disagreement by the stock's post-ranking β.[22]

Figure 5 reports the time-series of this disagreement measure. On the same plot, we also show a measure of uncertainty provided in Bloom et al. (2012). This measure consists in the

[21] We have also experimented with windsorizing at the 1% threshold and found similar results.

[22] An alternative measure is the dispersion of analyst forecasts of the earnings growth of the S&P 500 index. The problem with this top-down measure is that there are far fewer analysts forecasting this quantity, making it far less attractive when compared to our bottom-up measure.

cross-sectional dispersion of plant-level sales growth and is discussed extensively in Bloom et al. (2012). This measure captures the uncertainty underlying the economic fundamentals of the economy.

These two series are strongly correlated with our measure of aggregate disagreement on the overlapping sample (correlation of .57). The time-series of our aggregate disagreement and uncertainty measures peak during the 1981/82 recession, the dot-com bubble of the late 90s and the recent recession of 2008. When these fundamentals are more uncertain, there is more scope for disagreement among investors. Given how correlated these two series are, we focus on the aggregate disagreement measure for the most part but we also consider robustness checks using the longer sample period for the uncertainty measure which goes back to 1970.

4. Empirical Analysis

4.1. Predictions on Disagreement, Shorting and Share Turnover

We first start by testing the predictions of our model that are related to quantity as opposed to prices. Figure 6 highlights the role played by aggregate disagreement on the relationship between turnover, shorting and stock-level disagreement and β. For each of our 20 β portfolios, we compute the average of the stock-level short ratio (top panel), average stock-level dispersion in analyst earnings forecasts (middle panel) and average stock-level share turnover (bottom panel). These averages are calculated for high aggregate disagreement months (red dots) and low aggregate disagreement months (blue dots), where high (resp. low) aggregate disagreement months are defined as being in the top (resp. bottom) quartiles of in-sample aggregate disagreement. As shown in Figure 6, stock-level short-interest ratio, disagreement and turnover all increases with β; this relation is steeper in months with high aggregate disagreement relative to months with low aggregate disagreement.

Those results are confirmed by a simple regression analysis in Table 2, where we estimate the following equations:

$$\forall t \in [1981:12, 2010:12],\ \forall i \in [1,20]:$$
$$y_{it} = \alpha_t + \mu\beta_i + \nu Size_i + \quad \delta\beta_i \times \text{Agg. Dis.}_{t-1} + \rho Size_i \times \text{Agg. Dis.}_{t-1} + \epsilon_{it}$$

where y_{it} is the equal-weighted average short-interest ratio of stocks in portfolio i at date t (column 1 and 2), the equal-weighted average disagreement of stocks in portfolio i at date t (column 3 and 4) and the equal-weighted average turnover of stocks in portfolio i at date

t (column 5 and 6). This equation controls for size as a way to account for the existing heterogeneity in our 20 β portfolio that could potentially correlate with our dependent variables.

Column 1, 3 and 5 estimate the previous equation using simple pooled cross-sectional regressions where standard errors are clustered at the year/month level. Column 2, 4 and 6 use a Fama-MacBeth approach. This is a two-stage procedure. In the first-stage, we estimate the following cross-sectional regression every month:

$$\forall i \in [1, 20] : y_{it} = \alpha + \delta_t \beta_i + \rho_t Size_i + \epsilon_{it}$$

The second-stage projects the time-series of coefficients obtained in the first-stage (δ_t, ρ_t) against the time-series of aggregate disagreement, using a simple OLS estimation with robust standard errors.

$$\forall t \in [1981:12, 2010:12], \quad \begin{cases} \rho_t = \bar{\nu} + \bar{\rho} \text{Agg. Dis.}_{t-1} + \omega_t \\ \delta_t = \bar{\mu} + \bar{\delta} \text{Agg. Dis.}_{t-1} + \eta_t \end{cases}$$

Column 2, 4 and 6 reports the coefficients $(\bar{\delta}, \bar{\mu}, \bar{\rho}, \bar{\nu})$ estimated using this two-step procedure. The resulting $\bar{\delta}$ coefficients are very similar to the δ coefficients obtained using the simple pooled cross-sectional approach in column 1, 3 and 5.

The findings are broadly consistent with our model. Consider first the result regarding stock-level disagreement. We notice that even for the lowest possible value of aggregate disagreement (i.e. aggregate disagreement of 3.26), there is a strictly increasing relationship between stock-level disagreement and β ($1.1 \times 3.26 - 2.5 > 0$ in column 3). This validates an important modeling choice in our theory, which is to neglect idiosyncratic disagreement. Indeed, even in low aggregate disagreement months, high β stocks experience more stock-level disagreement than low β stocks. Thus, idiosyncratic disagreement appears to be positively correlated with β in the data. As we discussed in Section 2.4, this positive correlation should magnify the speculativeness of high β stocks. Moreover, we see that the relationship between stock-level disagreement and β becomes significantly steeper as aggregate disagreement increases. This is consistent with the basic premise of our analysis that β scales aggregate disagreement.

In column 1 and 2, we report that while the short-interest ratio typically increases with aggregate disagreement, this relation is significantly stronger in high aggregate disagreement months. This is consistent with our theory, since, mispricing is larger for high β stocks and hence arbitrageurs are on average more likely to short those stocks. However, mispricing is also increasing with λ, the aggregate disagreement parameter – so that there will be more

shorting by the arbitrageurs for high disagreement months. Note that these effects are quite sizable. Consider a one s.d. increase in aggregate disagreement (.90). It will lead to a relative increase in the short interest ratio of a $\beta = 1$ stock relative to a $\beta = 0$ stock of $.004 \times .90 = .0036$, which represents 20% of the s.d. of the short ratio.

Finally, column 5 and 6 shows that the same patterns hold for turnover: turnover in general increases with β, but much more so in months of high aggregate disagreement. This is consistent with our simple dynamic extension whereby in high disagreement states, more shares are being exchanged from one generation to the next due to the larger shorting activity by young group a investors.

4.2. Aggregate Disagreement and the Security Market Line

The Concavity of the Security Market Line

The second part of our empirical analysis examines how the Security Market Line is affected by aggregate disagreement. To this end, we first present in Figure 7 the empirical relationship between β and excess returns. For each of the 20 β portfolios in our sample, we compute the average excess forward return for high (red dots) and low (blue dots) disagreement months (defined as top vs. bottom quartile of aggregate disagreement). We do this using various horizons: 1-month (top left panel), 3-months (top-right panel), 6 months (bottom-left panel) and 12 months (bottom-right panel). While the relationship between excess forward returns and β is quite noisy at the 1 and 3 months horizon, two striking facts emerge at the 6 and 12 months horizons. First, the excess return/β relationship seems to be increasing for months with low aggregate disagreement. This is consistent with our theory whereby low aggregate disagreement means low or even no mispricing and hence a strictly upward sloping Security Market Line. Second, in months of high aggregate disagreement the excess returns/β relationship appears to exhibit the inverted-U shape predicted by the theory.

To evaluate the importance of this inverted-U shaped relationship in the data, we consider the following Fama-MacBeth regressions. In the first stage, we estimate, for each month in our sample the following cross-sectional regression:

$$\forall i \in [1, 20]: \quad r^e_{i,t\to t+k} = \alpha^k_t + \pi^k_t \beta_i + \phi^k_t \beta^2_i + \epsilon^k_{it}$$

where $r^e_{i,t\to t+k}$ is the k-months forward excess return of the i^{th} β portfolio at date t and β_i is the post-ranking β of portfolio i. This gives us a time-series of coefficient estimates $(\alpha_t, \pi_t, \phi_t)_{t\in[1981:12,2010:12]}$. We are especially interested in how the concavity in the excess return/β relationship evolves with aggregate disagreement. In fact, our theory predicts that the relationship should be more concave in high disagreement states than in low disagreement

states since the slope of the Security Market Line is constant in low disagreement state but decreasing in high disagreement states for sensible levels of persistence. To formally test this, we simply project this time-series of estimates onto our aggregate disagreement measure. Table 3 reports our findings. The first line corresponds to the ψ^k coefficient in the estimation of the following equation:

$$\phi_t^k = c^k + \psi^k \times \text{Agg. Dis.}_t + \omega_t^k$$

where $k \in \{1, 3, 6, 12\}$ corresponds to the various horizons used in the first-stage cross-sectional regressions. In this context, ψ^k is naturally interpreted as how the concavity in the excess returns/β relationship evolves with aggregate disagreement. In column 2, 4, 6 and 8, we control for other variables potentially affecting the shape of the security market line as well, i.e. we estimate an augmented version of the previous equation:

$$\phi_t^k = c^k + \psi^k \times \text{Agg. Dis.}_t + \tau^k X_t + \omega_t^k$$

where X_t includes the returns on HML (the long high book to market and short low book to market portfolios), SMB (the long small stock and short big stock portfolios), the PE ratio (the price to earnings ratio) and DP ratio (the dividend to price ratio), the TED spread, the one year inflation rate as well as the k year forward market return. When k is strictly larger than 1, we use Newey-West standard errors with $k-1$ lags to account for the overlapping returns in our dependent variable. When $k = 1$, we simply report Huber-White standard errors.

These additional control variables are meant to account for alternative explanations for our concave relationship between beta and returns. Our relationship, if anything, gets stronger when controlling for these covariates. Note that these covariates include forward market returns, which means that our results are not due to omitted market predictors of aggregate disagreement itself forecasting poor market returns as in Yu (2010). They also include HML, which means that our results are not driven by existing predictabilty patterns in the data such as value-growth.[23]

Moreover, our findings are not simply mechanical functions of disagreement forecasting returns in the cross-section as in Diether et al. (2002) since expected return is increasing in beta for low enough beta stocks even though the relationship between beta and stock level disagreement is monotonic as documented above. Finally, our aggregate disagreement and uncertainty measures are increasing in both up and down markets, which suggests then that

[23]Indeed, there is a value-growth effect for both high and low beta stocks, which suggests that the value-growth effect is uncorrelated with our beta patterns.

our findings are not simply a manifestation of high beta stocks doing poorly during bad times.

The results in Table 3 indicates that, at the 3, 6 and 12 months horizon, the Security Market Line is significantly more concave for months with high aggregate disagreement relative to months with low aggregate disagreement, i.e. ψ^k is negative and significantly different from 0 at the 5 or 1% level depending on the horizon and whether or not the control predictors are included in the regression. To better grasp the implication of this regression analysis on the shape of the security market line, we present, in Figure 8, the Security Market Line as predicted from the regression in Table 3 column 5 for two levels of aggregate disagreement: low (corresponding to the 10^{th} percentile of the aggregate disagreement distribution or $\lambda = 3.55$) and high (corresponding to the 90^{th} percentile of the aggregate disagreement distribution or $\lambda = 5.63$). As we see from this figure, the 6-months forward Security Market Line is predicted to be upward slopping for the low disagreement months and inverted-U shaped for the high disagreement months. While the excess return/β relationship is similar for low beta stocks in both high and low aggregate disagreement months, excess returns become significantly lower for high beta stocks in months of high aggregate disagreement relative to the low aggregate disagreement months: high β stocks are predicted to have 0 excess returns in high aggregate disagreement months at the 6-months horizon, as opposed to low disagreement months where excess returns are around 10%. These results are overall consistent with the predictions from our speculative asset pricing model.

The Slope of the Security Market Line

In Table 4, we present the results of a similar analysis omitting the quadratic term in the cross-sectional regressions. We are thus interested in the average slope of the Security Market Line and in how it relates with aggregate disagreement. We have shown that the theory predicts this average slope to be strictly higher in low disagreement states. To get at this result empirically, we again use a two-stages procedure, where we first regress in the cross-section:

$$\forall i \in [1, 20]: \quad r^e_{i,t\to t+k} = a^k_t + b^k_t \beta_i + \epsilon^k_{it}$$

and then project b^k_t onto aggregate disagreement at date t, as well as the control predictors X_t used in Table 3. The results in Table 4 reports the result from this two-step procedure. It shows that at the 6 and 12 months horizon, the slope of the Security Market Line is strictly lower in months with high aggregate disagreement. Again, this is a natural consequence of our model where high β stocks are speculative when aggregate disagreement is high, which

makes their expected returns go down.

4.3. Aggregate Uncertainty and the Security Market Line

Finally, we repeat our earlier empirical analysis using Bloom et al. (2012)'s economic uncertainty measure. The advantage of using this alternative measure is that it is available since 1970, which provides us with more statistical power to identify the role played by disagreement on asset prices. We think of this measure as a proxy for aggregate disagreement since when economic fundamentals are more uncertain, there is more scope for disagreement among investors. While economic uncertainty can have a *direct* impact on stock returns, i.e. not through the disagreement it generates, we expect this direct effect to be opposite to aggregate disagreement, and thus to bias us against finding significant results. However, the analysis using this measure yields result similar to those derived using our initial proxy for aggregate disagreement. Table 5 repeats the analysis of Table 3 using this uncertainty measure as our proxy for disagreement. At the 6 and 12 months horizons, the Security Market Line is again significantly more concave for months with higher uncertainty.

5. Conclusion

We show that incorporating the speculative motive for trade into asset pricing models yields strikingly different results from the risk-sharing or liquidity motives. High beta assets are more speculative since they are more sensitive to disagreement about common cash-flows. Hence they experience greater divergence of opinion and in the presence of short-sales constraint for some investors, they end up being over-priced relative to low beta assets. When aggregate disagreement is low, the risk-return relationship is upward sloping. As aggregate disagreement rises, the slope of the Security Market Line is piecewise constant, higher in the low beta range, and potentially negative for the high beta range. Empirical tests using security analyst disagreement and aggregate uncertainty measures are consistent with these predictions. We believe our simple and tractable model provides a plausible explanation for part of the high-risk, low-return puzzle.

References

Almazan, Andres, Keith C. Brown, Murray Carlson, and David A. Chapman, "Why constrain your mutual fund manager?," *Journal of Financial Economics*, August 2004, *73* (2), 289–321.

Baker, Malcolm, Brendan Bradley, and Jeffrey Wurgler, "Benchmarks as Limits to Arbitrage: Understanding the Low-Volatility Anomaly," *Financial Analysts Journal*, 2011, *67* (1), 1–15.

Barber, Brad M. and Terrance Odean, "Boys Will Be Boys: Gender, Overconfidence, And Common Stock Investment," *The Quarterly Journal of Economics*, February 2001, *116* (1), 261–292.

Barberis, Nicholas and Ming Huang, "Mental Accounting, Loss Aversion, and Individual Stock Returns," *Journal of Finance*, 2001, *56* (4), 1247–1292.

Black, Fischer, "Capital Market Equilibrium with Restricted Borrowing," *Journal of Business*, 1972, *4* (3), 444–455.

_ **, Michael C. Jensen, and Myron Scholes**, "The Capital Asset Pricing Model: Some Empirical Tests," in Michael C. Jensen, ed., *Studies in the Theory of Capital Markets*, New York: Praeger, 1972.

Blitz, David C. and Pim Van Vliet, "The Volatility Effect: Lower Risk without Lower Return," *Journal of Portfolio Management*, 2007, *34* (1), 102–113.

Bloom, Nicholas, "The Impact of Uncertainty Shocks," *Econometrica*, 05 2009, *77* (3), 623–685.

_ **, Max Floetotto, and Nir Jaimovich**, "Really Uncertain Business Cycles," *Stanford Working Paper*, 2012.

Brennan, Michael J., "Agency and Asset Pricing," University of California at Los Angeles, Anderson Graduate School of Management, Anderson Graduate School of Management, UCLA May 1993.

Campbell, John Y, Sanford J Grossman, and Jiang Wang, "Trading Volume and Serial Correlation in Stock Returns," *The Quarterly Journal of Economics*, 1993, *108* (4), 905–39.

Chen, Joseph, Harrison Hong, and Jeremy C. Stein, "Breadth of ownership and stock returns," *Journal of Financial Economics*, 2002, *66* (2-3), 171–205.

Cohen, Randolph B. Cohen Christopher Polk Randolph B., Christopher Polk, and Tuomo Vuolteenaho, "Money Illusion in the Stock Market: The Modigliani-Cohn Hypothesis," *The Quarterly Journal of Economics*, 2005, *120* (2), 639–668.

Cukierman, Alex and Paul Wachtel, "Differential Inflationary Expectations and the Variability of the Rate of Inflation: Theory and Evidence," *American Economic Review*, September 1979, *69* (4), 595–609.

Daniel, Kent D., David Hirshleifer, and Avanidhar Subrahmanyam, "Overconfidence, Arbitrage, and Equilibrium Asset Pricing," *Journal of Finance*, 2001, *56*, 921–965.

Delong, J Bradford, Andrei Shleifer, Lawrence Summers, and Mark Waldmann, "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance*, June 1990, *45* (2), 379–95.

Diether, Karl B., Christopher J. Malloy, and Anna Scherbina, "Differences of Opinion and the Cross Section of Stock Returns," *Journal of Finance*, October 2002, *57* (5), 2113–2141.

Fama, Eugene F and Kenneth R French, "The Cross-Section of Expected Stock Returns," *Journal of Finance*, June 1992, *47* (2), 427–65.

Frazzini, Andrea and Lasse Pedersen, "Betting Against Beta," Working Paper, New York University 2010.

Harrison, J Michael and David M Kreps, "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *The Quarterly Journal of Economics*, May 1978, *92* (2), 323–36.

Hong, Harrison and David Sraer, "Quiet Bubbles," Working Paper, Princeton University 2011.

_ **and Jeremy C. Stein**, "Disagreement and the Stock Market," *Journal of Economic Perspectives*, Spring 2007, *21* (2), 109–128.

_ **, Jose Scheinkman, and Wei Xiong**, "Asset Float and Speculative Bubbles," *Journal of Finance*, 06 2006, *61* (3), 1073–1117.

Jarrow, Robert A, "Heterogeneous Expectations, Restrictions on Short Sales, and Equilibrium Asset Prices," *Journal of Finance*, 1980, *35* (5), 1105–13.

Jegadeesh, Narasimhan and Sheridan Titman, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, March 1993, *48* (1), 65–91.

Kandel, Eugene and Neil D Pearson, "Differential Interpretation of Public Signals and Trade in Speculative Markets," *Journal of Political Economy*, August 1995, *103* (4), 831–72.

Koski, Jennifer Lynch and Jeffrey Pontiff, "How Are Derivatives Used? Evidence from the Mutual Fund Industry," *Journal of Finance*, 04 1999, *54* (2), 791–816.

Lakonishok, Josef, Andrei Shleifer, and Robert W Vishny, "Contrarian Investment, Extrapolation, and Risk," *Journal of Finance*, December 1994, *49* (5), 1541–78.

Lamont, Owen, "Go Down Fighting: Short Sellers vs. Firms," NBER Working Papers, National Bureau of Economic Research, Inc July 2004.

Lamont, Owen A., "Macroeconomic forecasts and microeconomic forecasters," *Journal of Economic Behavior & Organization*, July 2002, *48* (3), 265–280.

Mankiw, N. Gregory, Ricardo Reis, and Justin Wolfers, "Disagreement about Inflation Expectations," in "NBER Macroeconomics Annual 2003, Volume 18" NBER Chapters, National Bureau of Economic Research, Inc, 2004, pp. 209–270.

Merton, Robert C, "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance*, 1987, *42* (3), 483–510.

Miller, Edward M, "Risk, Uncertainty, and Divergence of Opinion," *Journal of Finance*, September 1977, *32* (4), 1151–68.

Morris, Stephen, "Speculative Investor Behavior and Learning," *The Quarterly Journal of Economics*, November 1996, *111* (4), 1111–33.

Odean, Terrance, "Do Investors Trade Too Much?," *American Economic Review*, December 1999, *89* (5), 1279–1298.

Scheinkman, Jose A. and Wei Xiong, "Overconfidence and Speculative Bubbles," *Journal of Political Economy*, December 2003, *111* (6), 1183–1219.

Sharpe, William, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance*, 1964, *XIX* (3), 425–442.

Yu, Jialin, "Disagreement and Portfolio Return Predictability," Working Papers, forthcoming *Journal of Financial Economics* 2010.

Tables and Figures

Table 1: Summary Statistics for Time Series Analysis

	Mean	Median	Std. Dev.	25^{th} p	75^{th} p.	Min	Max	Obs.
Aggregate Variable								
Agg. Dis	4.42	4.18	.897	3.72	4.77	3.26	7.33	349
Sales growth uncert.	8.023	7.616	1.540	6.864	8.712	5.909	13.220	486
Price/Earnings	22.41	19.00	15.26	15.32	25.59	7.48	123.79	349
Div./Price	0.026	0.024	0.011	0.017	0.033	0.011	0.062	349
SMB	0.14	-0.03	3.19	-1.61	1.84	-16.62	22.06	349
HML	0.37	0.34	3.13	-1.38	1.91	-12.87	13.88	349
TED	0.56	0.40	0.56	0.22	0.68	0.03	4.62	348
Inflation	3.07	3.00	1.42	2.23	3.92	-2.10	8.39	348
Portfolio Characteristics								
Stock Disagreement	3.898	3.613	1.321	3.094	4.395	0.000	13.813	6,967
Short-Interest Ratio	0.022	0.015	0.017	0.008	0.033	0.000	0.086	5,239
Turnover Ratio	0.734	0.548	0.504	0.382	0.966	0.053	3.080	7,260
1-month excess return	0.951	1.269	5.308	-1.476	3.606	-33.621	41.785	7,240
3-months excess return	2.899	3.212	10.215	-2.455	8.023	-52.574	79.062	7,200
6-months excess return	5.847	6.098	14.868	-1.624	13.037	-65.556	116.814	7,140
12-months excess return	12.274	12.887	21.978	-0.819	23.499	-72.850	150.968	7,020
β	0.901	0.844	0.386	0.583	1.141	0.359	1.811	7,060

Note: Agg. Dis. is the β-weighted average of stock level dispersion measured as the standard deviation of analyst forecasts on a stock. Sales growth uncert. is the cross-sectional dispersion of US plants sales growth taken from Bloom et al. (2011). Price/Earnings and Div./Price are the aggregate price-earning ratio and dividend-to-price ratio from Shiller's website. SMB and HML are the monthly returns on the HML and SML portfolios from French's website. TED is the TED spread and Inflation is the yearly inflation rate. Stock Disagreement is the portfolio equal-weighted average stock-level dispersion of analyst forecast. Turnover ratio is monthly volume normalized by number of shares outstanding. β is the post-ranking β.

Table 2: β portfolios characteristics and disagreement

	Short Interest		Stock Disagreement		Turnover	
	OLS	FM	OLS	FM	OLS	FM
$\beta\times$ Agg. Dis.$_{t-1}$	.004***	.0044***	1.1***	1.2***	.2***	.2***
	(7.6)	(7.7)	(19)	(18)	(12)	(12)
β	.0066***	.0054**	-2.5***	-2.6***	-.14**	-.15**
	(2.9)	(2.2)	(-10)	(-9.6)	(-2.1)	(-2)
Size $\times$ Agg. Dis.$_{t-1}$	.0016***	.00083***	.057**	-.052**	.02***	-.0069
	(7.8)	(5.08)	(2.4)	(-2.39)	(3.7)	(-1.20)
Size	-.0068***	-.0033***	-.6***	-.1652*	-.081***	.024
	(-8.3)	(-4.85)	(-6.2)	(-1.83)	(-3.7)	(1.0)
Observations	5,239	262	6,947	348	6,960	348
Adj. R^2	.91		.72		.9	

Notes: OLS and Fama-MacBeth (FM) estimation of monthly β portfolio characteristics on β, size and β and size interacted with aggregate disagreement. Aggregate disagreement is the β-weighted sum of stock level disagreement. The dependent variables are: the equal-weighted average monthly short interest ratio (column 1), the equal-weighted average stock-level disagreement (column 2), the equal-weighted turnover. β is the post-ranking portfolio β of the 20 β portfolio. Size is the logarithm of the average market capitalization of stocksin each 20-β portfolio. *, **, and *** means statistically different from zero at 10, 5 and 1% level of significance.

Table 3: Disagreement and concavity of the Security Market Line.

	Portfolio forward excess return							
	1-month		3-months		6-months		12 months	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\beta^2\times$ Aggregate Disagreement	-.728	-1.238**	-2.30**	-3.39***	-4.21**	-6.60***	-6.67**	-8.51
	(-1.56)	(-2.37)	(-2.12)	(-2.81)	(-2.45)	(-3.37)	(2.57)	(-3.65)
β^2	2.645	6.749**	8.68*	17.6***	16.09**	36.44***	26.47**	49.66***
	(1.39)	(2.53)	(1.96)	(2.90)	(2.27)	(3.70)	(2.50)	(3.82)
$\beta\times$ Aggregate Disagreement	1.00*	2.046***	2.877*	5.26***	4.54*	9.73**	6.11	6.32
	(1.68)	(2.73)	(1.79)	(2.74)	(1.76)	(3.21)	(1.35)	(1.51)
β	-3.113	-12.89***	-9.211	-32.50***	-14.25	-63.98***	-19.45	-56.01**
	(-1.25)	(-3.31)	(-1.36)	(3.38)	(-1.32)	(-3.70)	(-1.06)	(-2.22)
Controls	No	Yes	No	Yes	No	Yes	No	Yes
Observations	348	348	348	348	348	348	348	348

Notes: Fama-MacBeth estimation with Newey-West adjusted standard-errors allowing for 2 (column 3 and 4), 5 (column 5 and 6) and 11 lags (column 7 and 8). T-statistics are in parenthesis. The dependent variables are the 1/3/6/12 months forward equal-weighted return of 20 β portfolios. Aggregate disagreement is the β weighted average of stock-level disagreement measured as the dispersion of analysts forecasts on the long-run growth of EPS. Controls include the 1/3/6/12 months forward market return, HML, SMB, D/P, P/E, TED spread, past 12 months inflation. *, **, and *** means statistically different from zero at 10, 5 and 1% level of significance.

Table 4: Disagreement and the average slope of the security market line

	Slope of Security Market Line											
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Aggregate Disagreement	-.49	-.84*	-.49*	-1.8	-2.6**	-1.7**	-4.1*	-6.4***	-3.8**	-7.5*	-13***	-7.4***
	(-1.3)	(-1.8)	(-1.7)	(-1.4)	(-2)	(-2)	(-1.9)	(-3.2)	(-2.5)	(-1.9)	(-4.4)	(-3.1)
P/E		.031	.05**		.17**	.16***		.35***	.29***		.59***	.44***
		(.97)	(2.5)		(2.4)	(4.1)		(3.4)	(4.2)		(4)	(3.1)
D/P		3	-75***		-3.2	-232***		-98	-467***		-434	-873***
		(.071)	(-2.8)		(-.032)	(-3.4)		(-.52)	(-2.9)		(-1.3)	(-2.9)
HML		-.2*	-.12		-.84**	-.53**		-1***	-.56***		-1.2**	-1**
		(-1.7)	(-1.6)		(-2.6)	(-2.5)		(-3.1)	(-3.1)		(-2.1)	(-2.4)
SMB		-.035	-.12*		-.66***	-.4**		-1***	-.72***		-1.5**	-1.1***
		(-.3)	(-1.7)		(-2.9)	(-2.6)		(-3.6)	(-3.9)		(-2.6)	(-2.8)
Inflation		-.54	.35		-.47	1.3**		-.46	2.1**		-.41	1.8
		(-1.5)	(1.5)		(-.49)	(2.5)		(-.31)	(2.2)		(-.17)	(.99)
TED Spread		.75	1**		1.4	2.5***		4	4.9***		7.8*	8.2***
		(1)	(2.2)		(.61)	(2.9)		(1.4)	(3.3)		(1.7)	(3.1)
Forward Market Return			1.1***			1***			.96***			.84***
			(23)			(13)			(10)			(5.7)
Constant	2.3	4.4	.92	8.6	9.4	3.6	19**	23**	11	35**	56***	31**
	(1.3)	(1.6)	(.52)	(1.6)	(1.3)	(.76)	(2.1)	(2)	(1.2)	(2.2)	(2.7)	(2)
Observations	347	347	347	345	345	345	342	342	342	336	336	336
Adj. R^2												

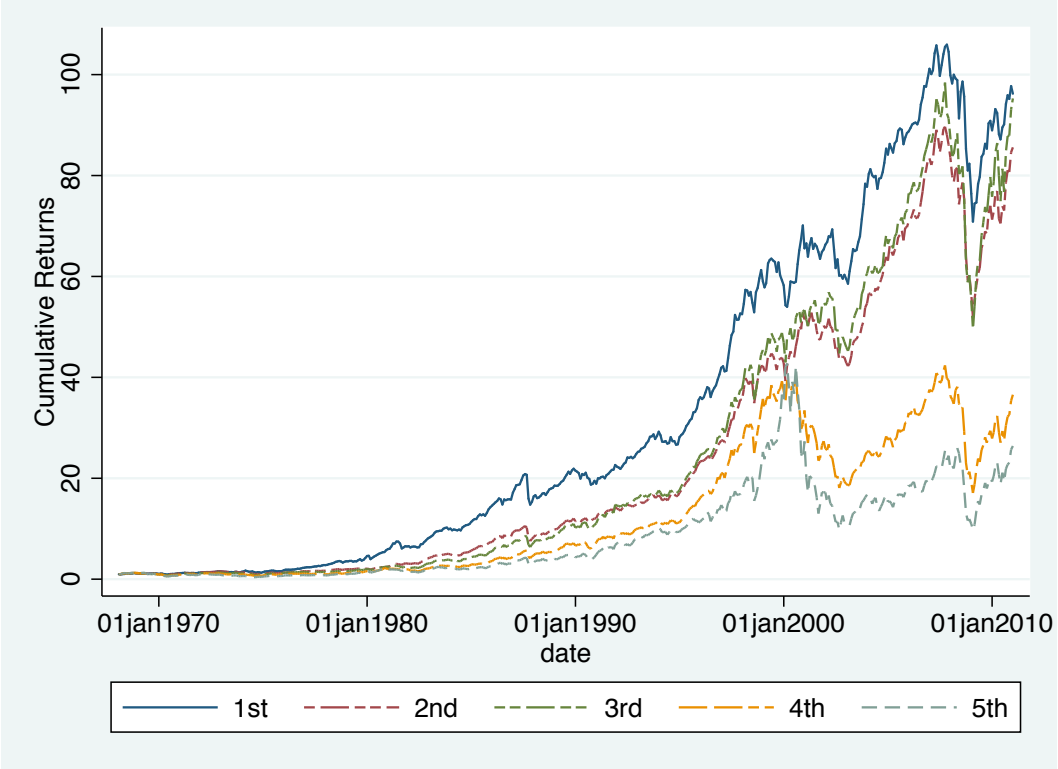
Notes: OLS estimation with Newey-West adjusted standard-errors allowing for 2/5/11 lags in column 4 to 12. Columns 1 to 3 use HuberWhite standard errors. T-statistics are in parenthesis. The dependent variable is the coefficient estimate of a monthly cross-sectional regression of the 20 beta-portfolio 1 month forward excess returns (columns 1 to 3), 3 months forward excess returns (columns 4 to 6), 6 months forward excess returns (columns 7 to 9) and 12 months forward excess returns (columns 10 to 12) on the portfolio beta. Aggregate disagreement is the β-weighted average of stock-level disagreement. Inflation rate is the past year CPI growth rate. D/P is the aggregate dividend to price ratio and P/E is the aggregate price to earnings ratio, both from Robert Shiller's website. HML and SMB are the returns on the HML and SMB portfolios from Ken French's website. *, **, and *** means statistically different from zero at 10, 5 and 1% level of significance.

Table 5: Aggregate Uncertainty (Bloom et al. (2012)) and concavity of the Security Market Line.

	Portfolio forward excess return							
	1-month		3-months		6-months		12 months	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\beta^2\times$ Uncert.	-.270	-.359	-.839	-.916	-2.058**	-2.513**	-4.80***	-4.896***
	(1.10)	(-.98)	(-1.64)	(-1.34)	(-2.17)	(-2.40)	(-3.23)	(-3.41)
β^2	1.84	4.03	6.026	9.249	15.41**	24.11**	37.96***	48.53***
	(1.0)	(1.42)	(1.57)	(1.60)	(2.14)	(2.57)	(3.26)	(3.22)
$\beta\times$ Uncert.	.424	.688	1.48	1.576	3.99**	3.826**	10.194***	6.492***
	(.91)	(1.23)	(1.37)	(1.41)	(2.01)	(2.32)	(4.35)	(2.83)
β	-2.64	-9.817**	-10.17	-22.32**	-29.38*	-47.63***	-79.85***	-76.749***
	(-.74)	(-2.19)	(-1.23)	(2.31)	(1.93)	(2.98)	(4.00)	(3.16)
Controls	No	Yes	No	Yes	No	Yes	No	Yes
Observations	476	342	476	342	476	342	476	342

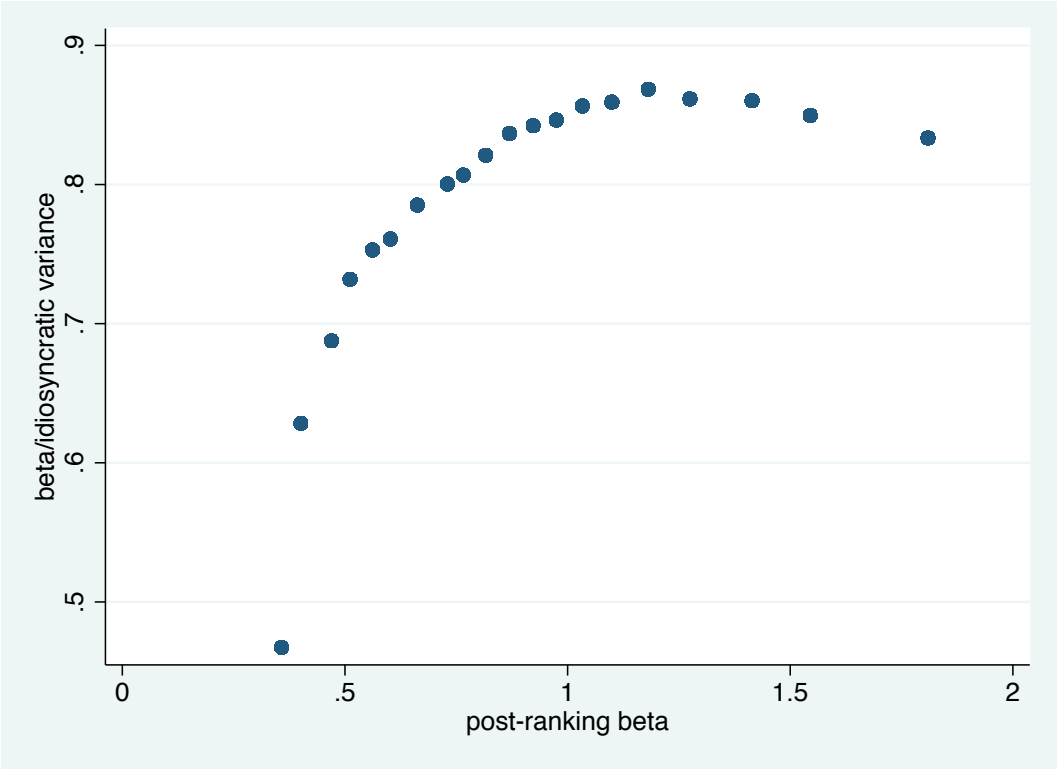
Notes: Fama-MacBeth estimation with Newey-West adjusted standard-errors allowing for 2 (column 3 and 4), 5 (column 5 and 6) and 11 lags (column 7 and 8). T-statistics are in parenthesis. The dependent variables are the 1/3/6/12 months forward equal-weighted return of 20 β portfolios.Uncert. corresponds to the aggregate uncertainty measure in Bloom et al. (2012) using the cross-sectional dispersion of US Census plants' sales growth. Controls include the 1/3/6/12 months forward market return, HML, SMB, D/P, P/E, TED spread, past 12 months inflation. *, **, and *** means statistically different from zero at 10, 5 and 1% level of significance.

Figure 1: High Risk, Low Return Puzzle



This figure plots the cumulative returns to one dollar invested on January 1968 to December 2010 for value-weighted portfolios of stocks sorted by a stock's one-year trailing beta computed using daily returns. 1st is the portfolio composed with stocks in the bottom quintile of beta. 5th is the portfolio composed with stocks in the top quintile of beta. Source: CRSP.

Figure 2: Median stock-level ratio of β to σ_ϵ^2 by post-ranking β.



This figure plots the median stock-level ratio of β to σ_ϵ^2 for our 20 β portfolios, as a function of the post-ranking β. See Section 3.2 for the computation of stock-level β and residual variance σ_ϵ^2 as well as for the construction of the β portfolios.

Figure 3: Timeline of the dynamic model in Section 2.5

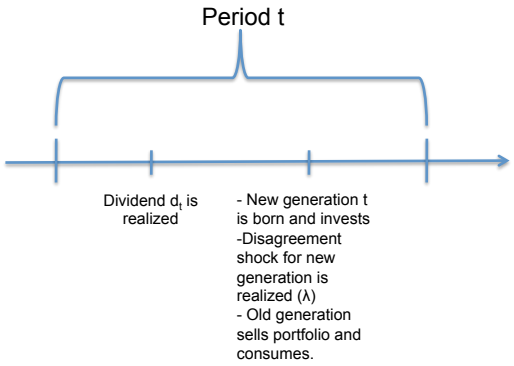
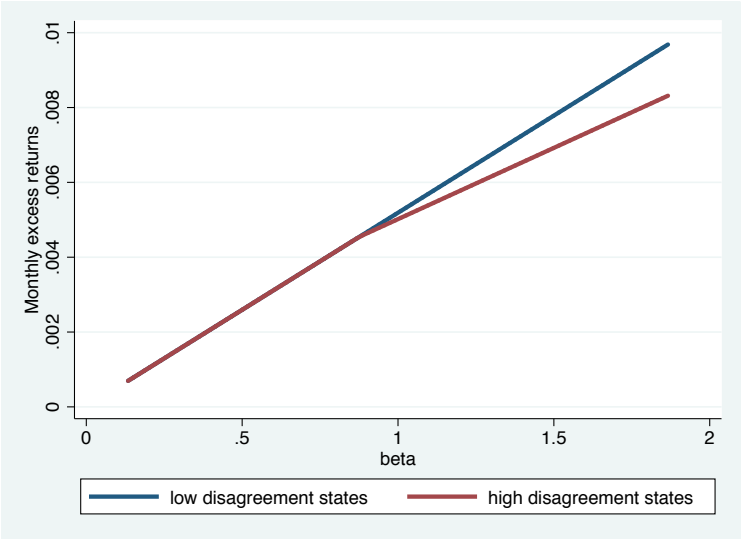
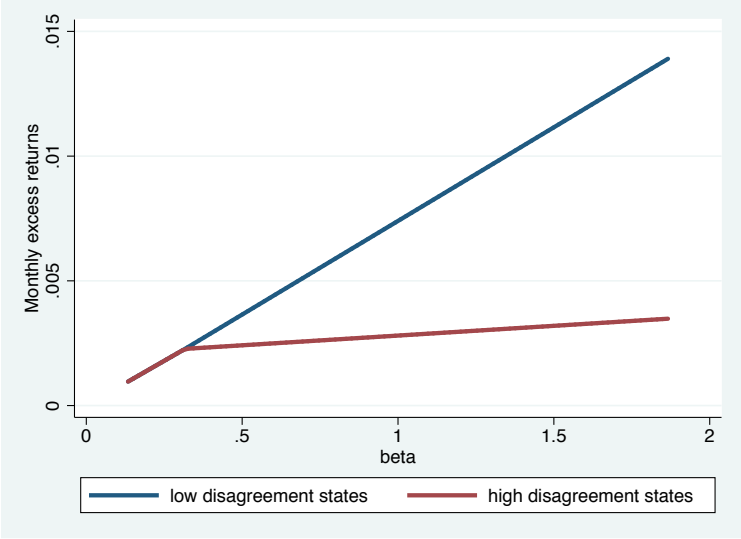


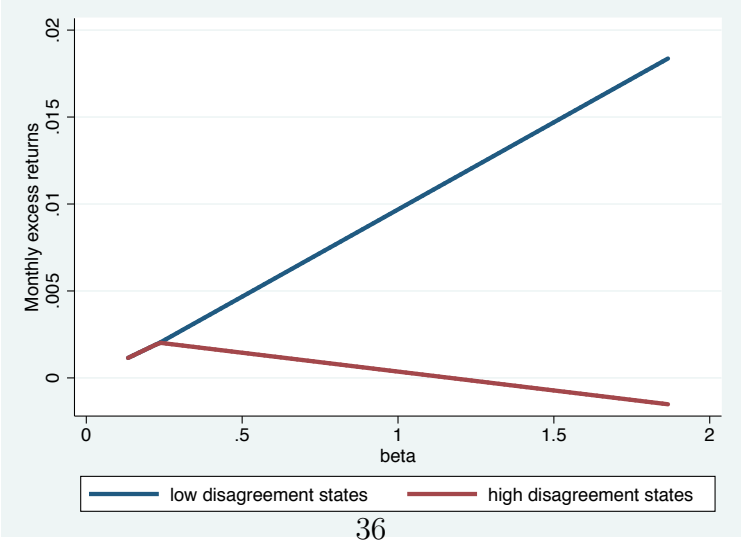
Figure 4: Model Calibration. See Section 2.6 for parameter values.



(a) $\lambda = .005$



(b) $\lambda = .01$



(c) $\lambda = .015$

Figure 5: Time-series of Aggregate Disagreement and Aggregate Uncertainty from Bloom et al. (2012)

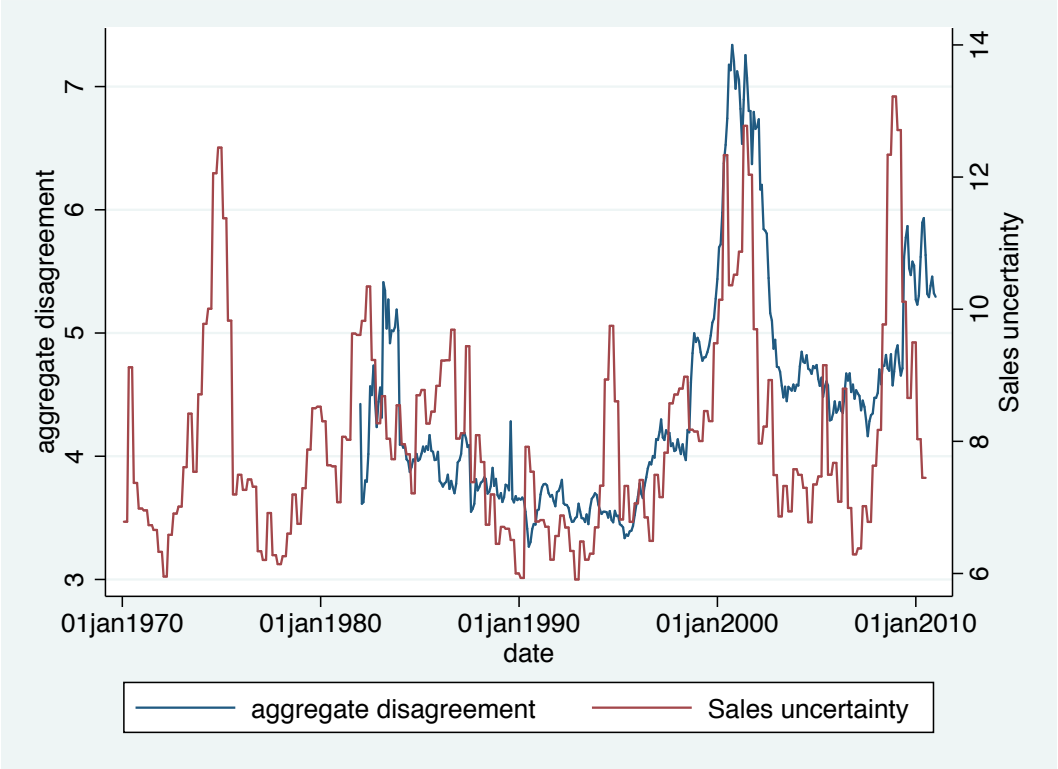
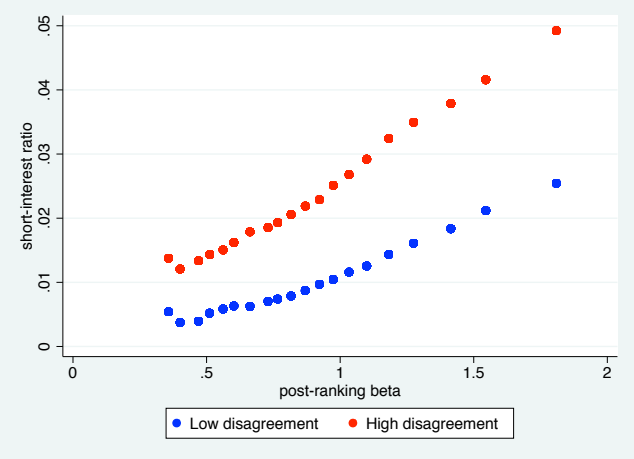
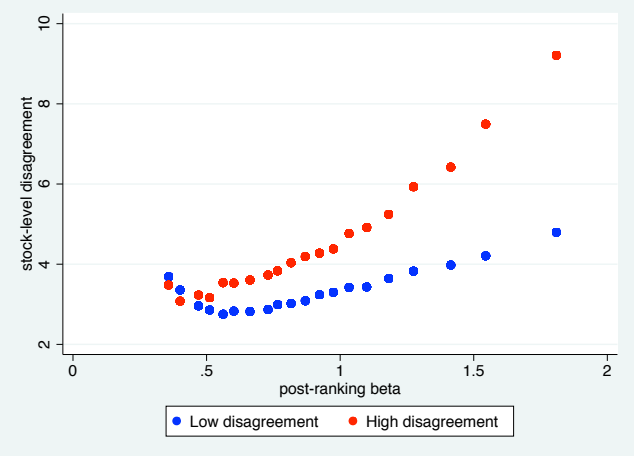


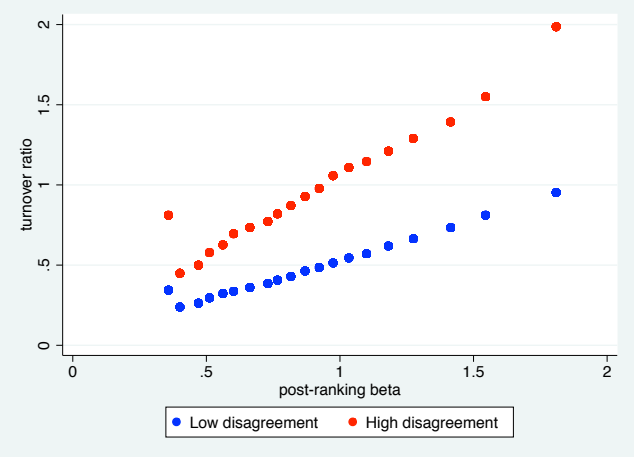
Figure 6: Equal-weighted average of analyst forecasts' dispersion, short interest ratio and share turnover for stocks by β (20 β portfolios) during low and high aggregate disagreement months.



(a) Short interest ratio

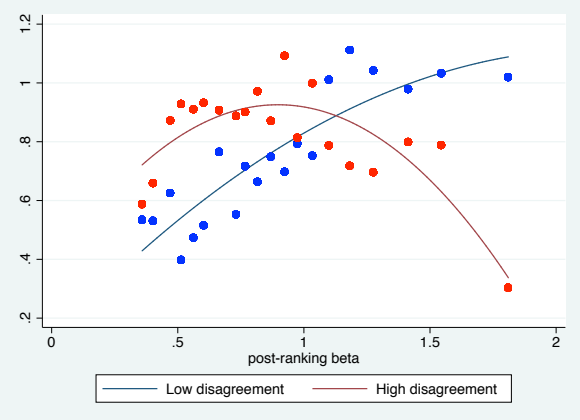


(b) Dispersion of analyst earnings forecasts

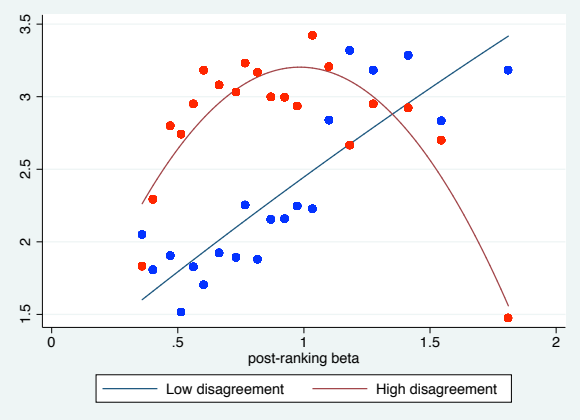


(c) Share turnover

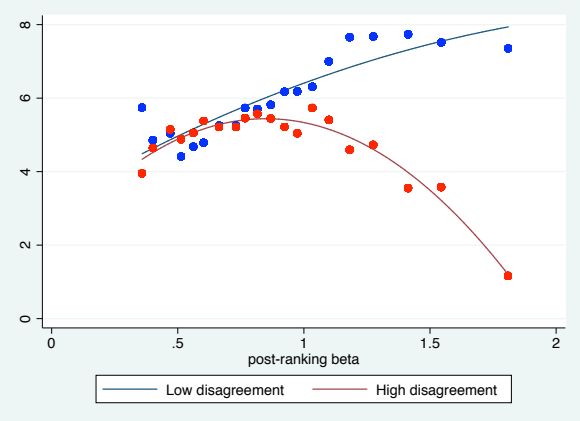
Figure 7: Average 1-month, 3-months, 6-months, and 12-months excess of risk-free returns for equal-weighted beta decile portfolios during low disagreement and high aggregate disagreement months.



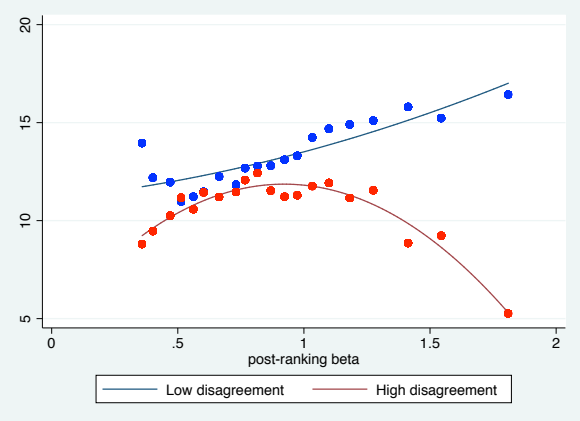
(a) 1 month equal-weighted return



(b) 3 months equal-weighted return

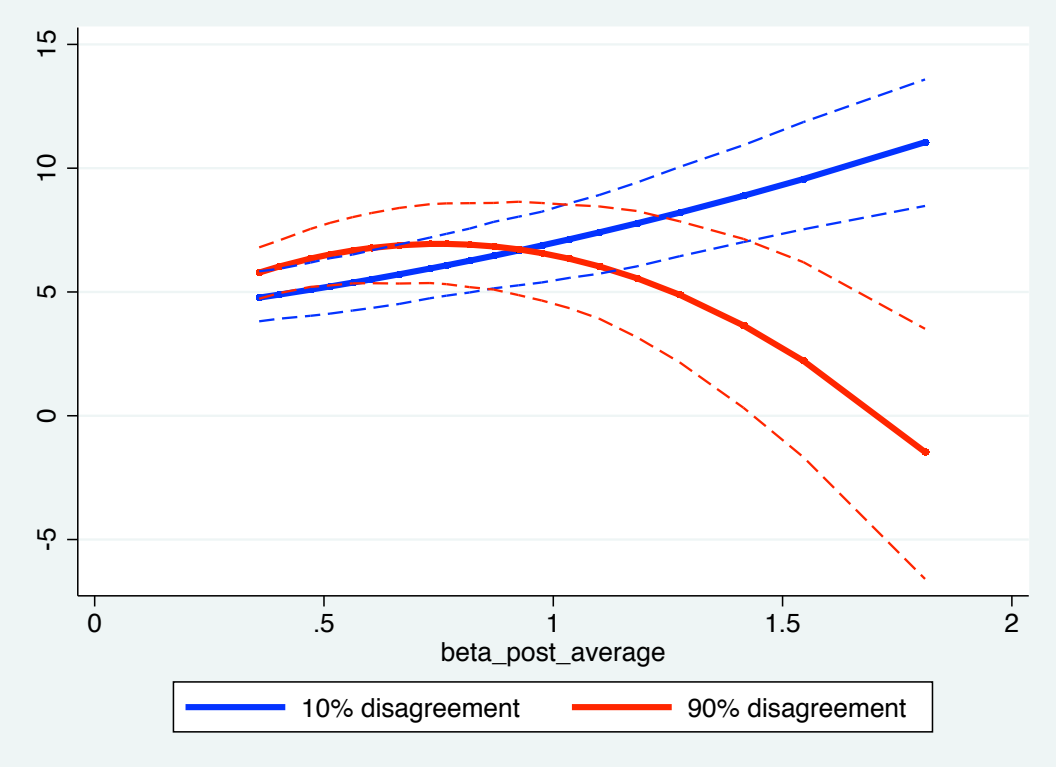


(c) 6 months equal-weighted return



(d) 12 months equal-weighted return

Figure 8: Predicted Security Market Line for 10% and 90% aggregate disagreement.



This figure provides the predicted relationship between expected returns and β from the estimation in Table 3 for two levels of disagreement, $\lambda = 3.55$ (10^{th} percentile) and $\lambda = 5.63$ (90^{th} percentile). Standard errors for the predicted values are bootstrapped.

A. For Online Publication: Appendix – Proofs

A.1. Proof of Theorem 1

Proof. In order to derive the conditions under which the proposed equilibrium is indeed an equilibrium (i.e. $\bar{i}$ is indeed the marginal asset), we need to derive the equilibrium holdings of group B investors:

$$\mu_i^{B,\star} = \begin{cases} \frac{1}{N} + b_i \left(\frac{\sigma_z^2 \left(\sum_{i \geq \bar{i}} \frac{b_i}{N} \right) - \lambda\gamma}{\sigma_\epsilon^2 + \sigma_z^2 \left(\sum_{i<\bar{i}} b_i^2 \right)} \right) & \text{for } i < \bar{i} \\ 0 \quad \text{for } i \geq \bar{i} \end{cases}$$

We are now ready to derive the conditions under which the proposed equilibrium is indeed an equilibrium. The marginal asset is asset $\bar{i}$ if and only if $\frac{\partial U^B}{\partial \mu_{\bar{i}}^B}(\mu^{B,\star}) < 0$ and $\mu_{\bar{i}-1}^B \geq 0$, where $\mu^{B,\star}$ is group B investors' holdings derived above. These conditions are easily shown to be equivalent to:

$$\frac{\sigma_z^2}{\gamma N} \sum_{k \geq \bar{i}} b_k + \frac{1}{\gamma N b_{\bar{i}-1}} \left(\sigma_\epsilon^2 + \sigma_z^2 \sum_{k<\bar{i}} b_k^2 \right) \geq \lambda > \frac{\sigma_z^2}{\gamma N} \sum_{k \geq \bar{i}} b_k + \frac{1}{\gamma N b_{\bar{i}}} \left(\sigma_\epsilon^2 + \sigma_z^2 \sum_{k<\bar{i}} b_k^2 \right)$$

Call $u_k = \frac{1}{\gamma N b_k} \left(\sigma_\epsilon^2 + \sigma_z^2 \left(\sum_{i<k} b_i^2 \right) \right) + \frac{\sigma_z^2}{\gamma N} \left(\sum_{i \geq k} \frac{b_i}{N} \right)$. Clearly, u_k is a strictly decreasing sequence as :

$$u_k - u_{k-1} = \frac{1}{\gamma} \underbrace{\left(\frac{1}{N b_k} - \frac{1}{N b_{k-1}} \right)}_{<0} \left(\sigma_\epsilon^2 + \sigma_z^2 \left(\sum_{i<k} b_i^2 \right) \right) < 0$$

Define $u_0 = +\infty$ and $u_{N+1} = 0$. Then the sequence $(u_i)_{i \in [0,N+1]}$ spans $\mathbb{R}^+$ and the marginal asset is simply defined as: $\bar{i} = \min \{k | \lambda > u_k\}$. We know that $\bar{i} > 0$ since $u_0 = +\infty$. If $\bar{i} = N+1$, then group B investors are long all assets and all the previous formula apply except that there is no asset such that $i \geq \bar{i}$. If $\bar{i} \in [1, n]$, then the equilibrium has the proposed structure, i.e. investors B are long only assets $i < \bar{i}$.

The proof in the main text considers the case $\bar{i} \geq 2$. The equilibrium is easily derived when $\bar{i} = 1$, i.e. when all assets are over-priced. In this case, $S = 0$ and we directly have:

$$\mu \bar{z} - (1+r) P_i = \frac{1}{\gamma} \left(b_i \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} - \theta \left(\lambda b_i - b_i \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} \right) \right)$$

. The equilibrium commands that $\lambda \leq u_N$, as stated in the Theorem.

The second part of the theorem characterizes overpricing. Overpricing for assets $i \geq \bar{i}$ is defined as the difference between the equilibrium price and the price that would prevail in the absence of heterogenous beliefs and short sales constraints ($\alpha = 0$). Overpricing is just simply equal to the speculative premium:

$$\forall i \geq \bar{i}, \quad \text{Overpricing}^i = \pi^i = \theta \left(b_i \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_z^2 \left(\sum_{i<\bar{i}} b_i^2 \right)} \left(\lambda - \frac{\sigma_z^2}{\gamma N} \sum_{k \geq \bar{i}} b_k \right) - \frac{\sigma_\epsilon^2}{\gamma N} \right)$$

By definition of the equilibrium, $\lambda > u_{\bar{i}}$. This directly implies: $\pi^i > 0$ so that assets $i \geq \bar{i}$ are in fact overpriced. That mispricing is increasing with the fraction of short-sales constrained investors α is direct as θ is a strictly increasing function of θ. That mispricing increases with b_i is also directly seen from the

definition of mispricing. Provided that $\bar{i} < N+1$, $\left(\lambda - \frac{\sigma_z^2}{\gamma N}\sum_{k\geq\bar{i}} b_k\right)$ has to be strictly positive, else the speculative premium would be strictly negative, which would violate the equilibrium condition. Then:

$$\forall j > i \geq \bar{i}, \quad \text{Overpricing}^j - \text{Overpricing}^i = \theta(b_j - b_i)\left(\lambda - \frac{\sigma_z^2}{\gamma N}\sum_{k\geq\bar{i}} b_k\right) > 0$$

The final part of the theorem characterizes the amount of shorting in the equilibrium. We first need to derive the equilibrium holdings of arbitrageurs. group a holdings need to satisfy the following first-order conditions:

$$\forall i \in [1,N], \quad \bar{z}b_i - P_i(1+r) = \frac{1}{\gamma}\left(b_i\sigma_z^2\left(\sum_{k=1}^{N}\mu_k^a b_k\right) + \mu_i^a\sigma_\epsilon^2\right)$$

Define $S^a = \sum_{k=1}^{N}\mu_k^a b_k$. Using the equilibrium pricing equation in equation 4 and equation 5, this first-order condition can be rewritten as:

$$\forall i \in [1,N], \quad b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} - \gamma\pi^i 1_{i\geq\bar{i}} = b_i\sigma_z^2 S^a + \mu_i^a\sigma_\epsilon^2$$

We multiply each of these equations by b_i and sum up the resulting equations for all $i \in [1,N]$ to obtain:

$$S^a = 1 - \gamma\frac{\sum_{k\geq\bar{i}} b_k\pi^k}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{k=1}^{N} b_k^2\right)}$$

We can now inject this expression for S^a in group a investors' first-order conditions derived above. This yields the following expression for group a investors' holdings of assets $i \in [1,N]$:

$$\forall i \in [1,N], \quad \mu_i^a = \frac{1}{N} - \frac{\gamma}{\sigma_\epsilon^2}\left(\pi^i 1_{\{i\geq\bar{i}\}} - b_i\sigma_z^2\frac{\sum_{k\geq\bar{i}} b_k\pi^k}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{k=1}^{N} b_k^2\right)}\right)$$

First remark that if $i < \bar{i}$, $\mu_i^a > 0$, so that arbitrageurs are long assets $i < \bar{i}$. Now consider the case $i \geq \bar{i}$. Notice from the expression of the speculative premium that:

$$\forall k, i \geq \bar{i}, \quad \pi_k + \frac{\theta\sigma_\epsilon^2}{\gamma N} = \frac{b_k}{b_i}\left(\pi^i + \frac{\theta\sigma_\epsilon^2}{\gamma N}\right)$$

Thus, multiplying the previous expression by b_k and summing over all $k \geq \bar{i}$:

$$\sum_{k\geq\bar{i}} b_k\pi_k + \frac{\theta\sigma_\epsilon^2}{\gamma N}\left(\sum_{k\geq\bar{i}} b_k\right) = \left(\sum_{k\geq\bar{i}} b_k^2\right)\left(\frac{\pi^i + \frac{\theta\sigma_\epsilon^2}{\gamma N}}{b_i}\right)$$

Thus:

$$\begin{aligned}
\pi^i - b_i\sigma_z^2\frac{\sum_{k\geq\bar{i}} b_k\pi^k}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)} &= \pi^i - b_i\frac{\sigma_z^2}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)}\left(\left(\sum_{k\geq\bar{i}} b_k^2\right)\left(\frac{\pi^i+\frac{\theta\sigma_\epsilon^2}{\gamma N}}{b_i}\right)-\frac{\theta\sigma_\epsilon^2}{\gamma N}\left(\sum_{k\geq\bar{i}} b_k\right)\right)\\
&= \pi^i\frac{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k<\bar{i}} b_k^2\right)}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)}-\frac{\sigma_z^2}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)}\frac{\theta\sigma_\epsilon^2}{\gamma N}\left(\sum_{k\geq\bar{i}} b_k^2-b_i\sum_{k\geq\bar{i}} b_k\right)\\
&= \frac{\theta\sigma_\epsilon^2}{\gamma}\left[b_i\left(\frac{\lambda\gamma-\frac{\sigma_z^2}{N}\left(\sum_{i\geq\bar{i}} b_i\right)}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}\right)-\frac{1}{N}\frac{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k<\bar{i}} b_k^2\right)}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)}\right.\\
&\qquad\left.-\frac{1}{N}\frac{\sigma_z^2}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{k=1}^N b_k^2\right)}\left(\sum_{k\geq\bar{i}} b_k^2-b_i\sum_{k\geq\bar{i}} b_k\right)\right]\\
&= \frac{\theta\sigma_\epsilon^2}{\gamma}\frac{1}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}\left[b_i\lambda\gamma-\frac{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}{N}\right]
\end{aligned}$$

We can now derive the actual holding of arbitrageurs on assets $i\geq\bar{i}$:

$$\forall i\geq\bar{i},\quad \mu_i^a=\frac{1}{N}+\theta\left(\frac{1}{N}-\frac{b_i\lambda\gamma}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}\right)=\frac{1+\theta}{N}-\theta\frac{b_i\lambda\gamma}{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}$$

First, notice that arbitrageurs holdings are decreasing with i since b_i increases strictly with i. There is at least one asset shorted by group a investors provided that $\mu_N^a<0$, which is equivalent to $\lambda>\hat{\lambda}=\frac{1+\theta}{\theta}\frac{\sigma_\epsilon^2+\sigma_z^2\left(\sum_{i=1}^N b_i^2\right)}{N}\frac{1}{\gamma b_N}$. Provided this is verified, there exists a unique $\tilde{i}\in[1,N]$ such that $\mu_i^a<0\Leftrightarrow i\geq\tilde{i}$. We know already that $\tilde{i}\geq\bar{i}$ since for $i<\bar{i}$, group a investors holdings are strictly positive. It is direct to see from the expression for group a investors holdings that provided that $i\geq\tilde{i}$, we have:

$$\frac{\partial|\mu_i^a|}{\partial\lambda}>0\quad\text{and}\quad\frac{\partial^2|\mu_i^a|}{\partial\lambda\partial b_i}>0$$

There is more shorting on high cash-flow beta assets. There is more shorting the larger is aggregate disagreement. The effect of aggregate disagreement on shorting is larger for high cash-flow beta assets. □

A.2. Proof of formula 6 for expected excess returns

Proof. Note that $\beta_i=\frac{b_i\sigma_z^2+\frac{\sigma_\epsilon^2}{N}}{\sigma_z^2+\left(\sum_{i=1}^N\frac{1}{N^2}\right)\sigma_\epsilon^2}$, so that $b_i=\beta_i\frac{\sigma_z^2+\left(\sum_{i=1}^N\frac{1}{N^2}\right)\sigma_\epsilon^2}{\sigma_z^2}-\frac{1}{N}\frac{\sigma_\epsilon^2}{\sigma_z^2}$. We can rewrite the pricing equations in terms of expected returns:

$$\begin{cases}\mathbb{E}[R_i]=\dfrac{b_i\sigma_z^2+\frac{\sigma_\epsilon^2}{N}}{\gamma}\quad\text{for } i<\bar{i}\\ \mathbb{E}[R_i]=\dfrac{b_i\sigma_z^2+\frac{1}{N}\sigma_\epsilon^2}{\gamma}-\pi^i\end{cases}$$

$$\begin{cases} \mathbb{E}[R_i] = \dfrac{b_i\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma} \quad \text{for } i < \bar{i} \\ \mathbb{E}[R_i] = \dfrac{b_i\sigma_z^2}{\gamma}\left(1 - \dfrac{\sigma_\epsilon^2}{\sigma_z^2}\theta\dfrac{\lambda\gamma - \sigma_z^2\left(\sum_{i\geq\bar{i}}\frac{b_i}{N}\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}\right) + \dfrac{\sigma_\epsilon^2}{\gamma N}(1+\theta) \quad \text{for } i \geq \bar{i} \end{cases}$$

Define $\kappa(\lambda) = \frac{\sigma_\epsilon^2}{\sigma_z^2}\frac{\lambda\gamma - \sigma_z^2\left(\sum_{i\geq\bar{i}}\frac{b_i}{N}\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}} b_i^2\right)}$. Using the definition of β_i, this can be rewritten as:

$$\begin{cases} \mathbb{E}[R_i] = \beta_i\dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma} \quad \text{for } i < \bar{i} \\ \mathbb{E}[R_i] = \beta_i\dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma}\left(1 - \kappa(\lambda)\theta\right) + \dfrac{\sigma_\epsilon^2}{\gamma N}\theta(1+\kappa(\lambda)) \quad \text{for } i \geq \bar{i} \end{cases}$$

□

A.3. Proof of Corollary 1

Proof. We can write the actual excess returns as:

$$\tilde{R}_i = \begin{cases} \beta_i\dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma} + \tilde{\eta}^i \quad \text{for } i < \bar{i} \\ \mathbb{E}[R_i] = \beta_i\dfrac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma}\left(1 - \kappa(\lambda)\theta\right) + \dfrac{\sigma_\epsilon^2}{\gamma N}\theta(1+\kappa(\lambda)) + \tilde{\eta}^i \quad \text{for } i \geq \bar{i} \end{cases}$$

where $\tilde{\eta_i} = b_i\tilde{u} + \tilde{\epsilon_i}$ and $\tilde{u} = \tilde{z} - \bar{z}$.

Using the fact that by definition, $\sum_i b_i = \sum_i \beta_i = N$, a cross-sectional regression of realized returns $\tilde{R}_i$ on β_i and a constant would deliver the following coefficient estimate:

$$\begin{aligned} \hat{\mu} &= \frac{\sum_{i=1}^N \beta_i\tilde{R}_i - \sum_{i=1}^N \tilde{R}_i}{\sum_{i=1}^N \beta_i^2 - N} \\ &= \frac{\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}}{\gamma}\left(1 + \frac{\gamma}{\sigma_z^2}\tilde{u} - \left(\frac{\sum_{i\geq\bar{i}}\beta_i^2 - \sum_{i\geq\bar{i}}\beta_i}{\sum_{i=1}^N \beta_i^2 - N}\right)\theta\kappa(\lambda)\right) + \left(\frac{\frac{1}{N}\sum_{i\geq\bar{i}}\beta_i - \frac{N-\bar{i}+1}{N}}{\sum_{i=1}^N \beta_i^2 - N}\right)\frac{\sigma_\epsilon^2}{\gamma}\theta\left(1+\kappa(\lambda)\right) \end{aligned}$$

Let $\frac{u_{\bar{i}-1}}{\gamma} > \lambda_1 > \lambda_2 > \frac{u_{\bar{i}}}{\gamma}$. Call $\bar{i}_1$ ($\bar{i}_2$) the threshold associated with disagreement λ_1 (resp. λ_2). We have that $\bar{i}_1 = \bar{i}_2 = \bar{i}$. Then:

$$\hat{\mu}(\lambda_1) - \hat{\mu}(\lambda_2) = -\frac{\theta\left(\kappa(\lambda_1) - \kappa(\lambda_2)\right)}{\sum_{i=1}^N \beta_i^2 - N}\left(\sigma_z^2\left(\sum_{i\geq\bar{i}}^N \beta_i^2 - \sum_{i\geq\bar{i}}^N \beta_i\right) + \frac{\sigma_\epsilon^2}{N}\sum_{i\geq\bar{i}}^N \left(\beta_i - 1\right)^2\right)$$

We show that $\sum_{i\geq\bar{i}}\beta_i^2 \geq \sum_{i\geq\bar{i}}\beta_i$. Call $\beta_i = 1 + y_i$ with y_i such that $\sum y_i = 0$. Then: $\sum_{i=1}^N \beta_i^2 = N + \underbrace{2\sum_{i=1}^N y_i}_{=0} + \sum_{i=1}^N y^2 > N = \sum_{i=1}^N \beta_i$. Thus, the relationship is true for $\bar{i} = 0$. Now assume it is true for

$\bar{i} = k$. We have: $\sum_{i\geq k+1}\beta_i^2 - \sum_{i\geq k+1}\beta_i = \sum_{i\geq k}\beta_i^2 - \sum_{i\geq k}\beta_i + \beta_k - \beta_k^2$. Either $\beta_k > 1$ in which case it is evident that $\sum_{i\geq k+1}\beta_i^2 - \sum_{i\geq k+1}\beta_i > 0$ as $\beta_i > 1$ for $i \geq k$. Or $\beta_k \leq 1$ in which case $\beta_k - \beta_k^2 > 0$ and using the recurrence assumption, $\sum_{i\geq k+1}\beta_i^2 - \sum_{i\geq k+1}\beta_i > 0$. Thus, $\hat{\mu}(\lambda_1) - \hat{\mu}(\lambda_2) < 0$. Moreover, we show now that $\hat{\mu}(\lambda)$ is continuous at u_i, for all $i \in [1, N]$. When $\lambda = u_i^+$, we have $\bar{i} = i$. When $\lambda = u_i^-$, we have $\bar{i} = i + 1$. First, notice that $\kappa(\lambda)$ is continuous at u_i. Indeed:

$$\kappa(u_i^-) = \kappa(u_i^+) = \frac{\sigma_\epsilon^2}{\sigma_z^2}\frac{1}{N\gamma b_i}$$

Thus:

$$\begin{aligned}\hat{\mu}\left(u_i^+\right) - \hat{\mu}\left(u_i^-\right) &= \frac{\theta}{\gamma}\frac{\beta_{\bar{i}} - 1}{\sum_{i=1}^N \beta_i^2 - N}\left(\beta_i\kappa(u_i)\left(\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) - \frac{\sigma_\epsilon^2}{N}\left(1 + \kappa(u_i)\right)\right) \\ &= \frac{\theta}{\gamma}\frac{\beta_{\bar{i}} - 1}{\sum_{i=1}^N \beta_i^2 - N}\left(b_i\sigma_z^2\kappa(u_i) - \frac{\sigma_\epsilon^2}{N}\right) = 0\end{aligned}$$

Thus $\hat{\mu}$ is continuous and strictly decreasing over $]u_{i+1}, u_i[$, so that it is overall strictly decreasing with aggregate disagreement λ. We can also easily show that the slope of the security market line, $\hat{\mu}$, is strictly decreasing with θ, the fraction of short-sales constrained investors in the model. Indeed (noting that the thresholds u_i are independent of θ):

$$\frac{\partial\hat{\mu}}{\partial\theta} = -\frac{\kappa(\lambda)}{\gamma\left(\sum_{i=1}^N \beta_i^2 - N\right)}\left(\frac{\sigma_\epsilon^2}{N}\sum_{i\geq\bar{i}}\left(\beta_i - 1\right)^2 + \sigma_z^2\left(\sum_{i\geq\bar{i}}\beta_i^2 - \sum_{i\geq\bar{i}}\beta_i\right)\right) < 0$$

As we have already shown that: $\sum_{i\geq\bar{i}}\beta_i^2 - \sum_{i\geq\bar{i}}\beta_i$. Finally, going back to the expression for $\hat{\mu}(\lambda_1) - \hat{\mu}(\lambda_2)$, we see that this difference can be expressed as $-C\theta$ with $C > 0$, so that it is clearly increasing with θ. Thus, when α increases, θ increases, and the difference between $\hat{\mu}(\lambda_1)$ and $\hat{\mu}(\lambda_2)$ decreases. Because this difference is strictly negative, this means that the gap between the two slopes becomes wider. □

A.4. Proof of Theorem 2

Proof. We first consider the case where $\tilde{\lambda}_t = 0$. There is no disagreement among investors so all investors are long all assets $i \in [1, N]$. There is thus a unique first order-condition for all investors' type – for all $j \in [1, N]$ and $k = a$, A or B:

$$b_j\bar{z} - (1+r)P_t^j(0) + \mathbb{E}_{\tilde{\lambda}_{t+1}}[P_{t+1}^j(\tilde{\lambda}_{t+1})|0] = \frac{1}{\gamma}\left(b_j\sigma_z^2\sum_{i\leq N}\mu_i^k(0)b_i + \mu_j^k(0)\sigma_\epsilon^2 + \rho(1-\rho)\Delta P_{t+1}^j\left(\sum_{i\leq N}\mu_i^k(0)\left(\Delta P_{t+1}^i\right)\right)\right)$$

Summing up this equation across investors' types, using the market clearing condition and dropping the time subscript leads to:

$$b_j\bar{z} - (1+r)P^j(0) + \mathbb{E}_{\tilde{\lambda}_{+1}}[P^j(\tilde{\lambda}_{+1})|\tilde{\lambda} = 0] = \frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\Delta P^j\left(\sum_{i\leq N}\frac{\left(\Delta P^i\right)}{N}\right)\right) \tag{7}$$

Consider now the case where $\lambda_t = \lambda$. Importantly, note that investors disagree on the expected value of the aggregate factor $\tilde{z}$, but they agree on the expected value of asset i's resale price $\mathbb{E}_{\tilde{\lambda}_{t+1}}[P^j_{t+1}(\tilde{\lambda}_{t+1})]$. This is because investors agree to disagree, so they recognize the existence in the next generation of investors with heterogeneous beliefs – and in particular with beliefs different from theirs. However, they nevertheless evaluate the $t+1$ expected dividend stream differently. We proceed as in the static model. We assume there is a marginal asset $\bar{i}$, such that there are no binding short-sales constraints for assets $j < \bar{i}$ and strictly binding short-sales constraints for assets $j \geq \bar{i}$. We check ex post the conditions under which this is indeed an equilibrium. Under the proposed equilibrium structure, the first-order condition of the three groups of investors born at date t write, for assets $j < \bar{i}$:

$$b_j(\bar{z} + \lambda^k_t) - (1+r)P^j_t(\lambda) + \mathbb{E}_{\tilde{\lambda}_{t+1}}[P^j_{t+1}(\tilde{\lambda}_{t+1})|\tilde{\lambda}_t = \lambda] =$$
$$\frac{1}{\gamma}\left(b_j\sigma^2_z\sum_{i\leq N}\mu^k_i(\lambda)b_i + \mu^k_j(\lambda)\sigma^2_\epsilon + \rho(1-\rho)\Delta P^j_{t+1}\left(\sum_{i\leq N}\mu^k_i(\lambda)\left(\Delta P^i_{t+1}\right)\right)\right)$$

Dropping the time subscript, summing up across investors types and using the market clearing condition leads to:

$$\forall j < \bar{i}, \quad b_j\bar{z} - (1+r)P^j(\lambda) + \mathbb{E}_{\tilde{\lambda}_{+1}}[P^j(\tilde{\lambda}_{+1})|\lambda] = \frac{1}{\gamma}\left(b_j\sigma^2_z + \frac{\sigma^2_\epsilon}{N} + \rho(1-\rho)\Delta P^j_{t+1}\sum_{i\leq N}\frac{\Delta P^i_{t+1}}{N}\right) \tag{8}$$

Subtracting equation (7) from equation (8) leads to:

$$\forall j < \bar{i}, \quad -(1+r)\Delta P^j + \rho\Delta P^j - (1-\rho)\Delta P^j = 0 \Leftrightarrow P^j(\lambda) = P^j(0)$$

Thus, for all $j < \bar{i}$, $\Delta P^j = 0$. The payoff of assets below $\bar{j}$ is not sufficiently exposed to aggregate disagreement to make pessimist investors willing to go short. Hence, even in the high disagreement state, these assets experience no mispricing and in particular, their price is independent of the realization of aggregate disagreement. Aggregate disagreement thus only creates resale price risk on these assets that experience binding short-sales constraint in the high aggregate disagreement states, i.e. the high b assets with $i \geq \bar{i}$. We now turn to the assets with binding short-sales constraints in the high disagreement states, i.e. the assets $j > \bar{i}$. For these assets, we know that under the proposed equilibrium $\mu^B_j(\lambda) = 0$ and we have the following first-order conditions for HF and optimist MFs:

$$\begin{cases} b_j(\bar{z}+\lambda) - (1+r)P^j_t(\lambda) + \mathbb{E}_{\tilde{\lambda}_{t+1}}[P^j_{t+1}(\tilde{\lambda}_{t+1})|\lambda] = \frac{1}{\gamma}\left(b_j\sigma^2_z\sum_{i\leq N}\mu^A_i(\lambda)b_i + \mu^A_j(\lambda)\sigma^2_\epsilon + \rho(1-\rho)\Delta P^j_{t+1}\left(\sum_{i\leq N}\mu^A_i(\lambda)\Delta P^i_{t+1}\right)\right) \\ b_j\bar{z} - (1+r)P^j_t(\lambda) + \mathbb{E}_{\tilde{\lambda}_{t+1}|\lambda}[P^j_{t+1}(\tilde{\lambda}_{t+1})] = \frac{1}{\gamma}\left(b_j\sigma^2_z\sum_{i\leq N}\mu^a_i(\lambda)b_i + \mu^a_j(\lambda)\sigma^2_\epsilon + \rho(1-\rho)\Delta P^j_{t+1}\left(\sum_{i\leq N}\mu^a_i(\lambda)\Delta P^i_{t+1}\right)\right) \end{cases}$$

Summing up these equations across investors' types, using the market clearing conditions and dropping the time subscript lead to:

$$\frac{\alpha}{2}b_j\lambda + (1-\frac{\alpha}{2})\left(b_j\bar{z} - (1+r)P^j(\lambda) + \mathbb{E}_{\tilde{\lambda}_{t+1}}[P^j_{t+1}(\tilde{\lambda}_{t+1})|\lambda]\right) =$$

$$\frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\Delta P^j\Gamma - \frac{\alpha}{2}b_j\sigma_z^2\underbrace{\sum_{i<\bar{i}}\mu_i^B(\lambda)b_i}_{S^1} - \frac{\alpha}{2}\rho(1-\rho)\Delta P^j\underbrace{\sum_{i<\bar{i}}\mu_i^B(\lambda)\left(\Delta P^i\right)}_{S^2=0}\right)$$

Where $\Gamma = \sum_{i\geq\bar{i}}\frac{\Delta P^i}{N}$. In the previous equation, $S^2 = 0$ since for all $i < \bar{j}$, $\Delta P^i = 0$. To recover S^1, we use B-investors' first-order condition on assets $j < \bar{i}$, the equilibrium prices derived above for assets $j < \bar{i}$ and the fact that for all $i < \bar{j}$, $\Delta P^j = 0$. This leads to the following equation :

$$\forall j < \bar{i}, \quad b_j\sigma_z^2\underbrace{\sum_{i\leq N}\mu_i^B b_i}_{S^1} + \mu_j^B\sigma_\epsilon^2 = -\lambda\gamma b_j + b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N}$$

Multiplying the previous expression by b_i for $i < \bar{i}$ and summing up the equations over i gives a formula for S^1 similar to the static model:

$$S^1 = 1 - \frac{\sigma_\epsilon^2\left(\sum_{i\geq\bar{i}}\frac{b_i}{N}\right) + \lambda\gamma\left(\sum_{i<\bar{i}}b_i^2\right)}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}}b_i^2\right)}$$

This allows us to derive the excess return on assets $j \geq \bar{i}$:

$$\underbrace{b_j\bar{z} - (1+r)P^j(\lambda) + \mathbb{E}_{\tilde{\lambda}_{t+1}}[P^j_{t+1}(\tilde{\lambda}_{t+1})|\lambda]}_{\text{Excess Return}} =$$

$$\underbrace{\frac{1}{\gamma}\left(b_j\sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + (1+\theta)\rho(1-\rho)\Delta P^j\Gamma\right)}_{\text{Risk Premium}} - \underbrace{\theta\left(b_j\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_z^2\left(\sum_{i<\bar{i}}b_i^2\right)}\left(\lambda - \frac{\sigma_z^2}{\gamma N}\sum_{k\geq\bar{i}}b_k\right) - \frac{\sigma_\epsilon^2}{\gamma N}\right)}_{\pi^j=\text{speculative premium}}$$

Note that the risk premium embeds a term that reflects the resale price risk of high b assets. Subtracting equation (7) from the previous equation yields, for all $j \geq \bar{i}$:

$$-(1+r)\Delta P^j + (2\rho-1)\Delta P^j = -\pi^j + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma\Delta P^j \Rightarrow \left((1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma\right)\Delta P^j = \pi^j$$

Remember that $\Gamma = \sum_{i\geq\bar{i}}\frac{\Delta P^i}{N}$. We can thus obtain a formula for Γ by adding up the previous equations for all $j \geq \bar{i}$:

$$\left((1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma\right)\Gamma = \frac{1}{N}\sum_{j\geq\bar{i}}\pi^j$$

There is a unique $\Gamma^+ > 0$ which satisfies the previous equation, call it Γ^+:

$$\Gamma^\star = \Gamma^+ = \frac{-(1+r)+(2\rho-1)+\sqrt{\left((1+r)-(2\rho-1)\right)^2+\frac{4}{N}\frac{\theta\rho(1-\rho)}{\gamma}\sum_{j\geq\bar{i}}\pi^j}}{2\frac{\theta\rho(1-\rho)}{\gamma}}$$

There is also a unique $\Gamma < 0$ which satisfies the quadratic equation defining Γ:

$$\Gamma^- = \frac{-(1+r)+(2\rho-1)-\sqrt{\left((1+r)-(2\rho-1)\right)^2+\frac{4}{N}\frac{\theta\rho(1-\rho)}{\gamma}\sum_{j\geq\bar{i}}\pi^j}}{2\frac{\theta\rho(1-\rho)}{\gamma}}$$

And for $j \geq \bar{i}$: $\Delta P^j = \frac{\pi^j}{1+r-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}$ For the equilibrium to exist, it needs to be that for each asset $j \geq \bar{i}$, the pessimists do not want to hold the asset j, i.e. the marginal utility of holding assets $j \geq \bar{i}$ at the optimal holding is 0. This is equivalent to:

$$\forall j \geq \bar{i} \quad (b_j(\bar{z}-\lambda)-(1+r)P^j(\lambda)+\rho P^j(\lambda)+(1-\rho)P^j(0)-\frac{1}{\gamma}b_j\sigma_z^2\underbrace{\sum_{j<\bar{i}}\mu_i^B b_i}_{=S} < 0$$

We have:

$$\begin{aligned}
& (b_j(\bar{z}-\lambda)-(1+r)P^j(\lambda)+\rho P^j(\lambda)+(1-\rho)P^j(0)-\frac{1}{\gamma}b_j\sigma_z^2 S \\
= & -b_j\lambda+\frac{1}{\gamma}\left(b_j\sigma_z^2+\frac{\sigma_\epsilon^2}{N}+\rho(1-\rho)\Delta P^j\Gamma\right)-\pi^j+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma\Delta P^j-\frac{1}{\gamma}b_j\sigma_z^2 S \\
= & -\frac{\pi^j}{\theta}-\pi^j+(1+\theta)\frac{\rho(1-\rho)}{\gamma}\Gamma\Delta P^j \\
= & \frac{1+\theta}{\theta}\pi^j\left(\frac{\frac{\theta\rho(1-\rho)}{\gamma}\Gamma}{(1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma}-1\right) \\
= & -\frac{1+\theta}{\theta}\frac{(1+r)-(2\rho-1)}{(1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma}\times\pi^j
\end{aligned}$$

Consider first the case where $\Gamma^\star = \Gamma^- < 0$. We know that:

$$\theta\rho(1-\rho)\frac{\Gamma^-}{\gamma}+(1+r)-(2\rho-1) = \frac{(1+r)-(2\rho-1)-\sqrt{\left((1+r)-(2\rho-1)\right)^2+\frac{4}{N}\frac{\theta\rho(1-\rho)}{\gamma}\sum_{j\geq\bar{i}}\pi^j}}{2\frac{\theta\rho(1-\rho)}{\gamma}} < 0$$

Thus, if $\Gamma^\star = \Gamma^-$, then $-\frac{1+\theta}{\theta}\frac{(1+r)-(2\rho-1)}{(1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma} > 0$ so that it has to be that for all $j \geq \bar{i}$, $\pi^j < 0$. Thus: $\sum_{j\geq\bar{i}}\pi^j < 0$, so that

$$\left((1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma^-\right)\Gamma^- < 0$$

However, the previous expression is > 0 since $\Gamma^- < 0$ and $(1+r)-(2\rho-1)+\frac{\theta\rho(1-\rho)}{\gamma}\Gamma^- < 0$ as well. Thus, we can't have $\Gamma^\star = \Gamma^-$. Since $\Gamma^\star > 0$, we have from the previous equilibrium condition that necessarily, for all $j \geq \bar{i}$, $\pi^j > 0$. Similarly, it is direct to show that for pessimists to have strictly positive holdings of assets $j \bar{-} 1$, a necessary and sufficient condition is that $\pi^{\bar{j}-1} < 0$. This leads to the same condition for the

existence of the equilibrium than in the static case, i.e.: $u_{\bar{i}} < \lambda \leq u_{\bar{i}-1}$. Since $\Delta P^j = 0$ for $j < \bar{i}$, we have that for all $j < \bar{i}$:

$$\mathbb{E}[R^j(\lambda)] = \mathbb{E}[R^j(0)] = b_j \bar{z} - rP^j(\lambda) = b_j \bar{z} - rP^j(0) = \frac{1}{\gamma}\left(b_j \sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right)$$

For $j \geq \bar{i}$, however:

$$\begin{aligned}
\mathbb{E}[R^j(0)] &= b_j \bar{z} - (1+r)P^j(0) + \rho P^j(0) + (1-\rho)P^j(\lambda) \\
&= \frac{1}{\gamma}\left(b_j \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j\right)
\end{aligned}$$

The extra-term is the risk-premium required by investors for holding stocks which are sensitive to disagreement and are thus exposed to changes in prices coming from changes in the aggregate disagreement state variable. Of course, in the data, since ρ is very close to 1, this risk premium is going to be quantitatively small. However, the intuition here is that high b stocks have low prices in the low disagreement states for two reasons: (1) they are exposed to the aggregate risk $\tilde{z}$ (2) they are exposed to changes in aggregate disagreement $\tilde{\lambda}$. And finally:

$$\begin{aligned}
\mathbb{E}[R^j(\lambda)] &= b_j \bar{z} - (1+r)P^j(\lambda) + \rho P^j(\lambda) + (1-\rho)P^j(0) \\
&= \frac{1}{\gamma}\left(b_j \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j\right) - \pi^j \\
&\quad + \theta\frac{\rho(1-\rho)}{\gamma}\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j \\
&= \frac{1}{\gamma}\left(b_j \sigma_z^2 + \frac{\sigma_\epsilon^2}{N} + \rho(1-\rho)\frac{\Gamma^\star}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j\right) - \frac{1+r-(2\rho-1)}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j
\end{aligned}$$

Thus, for assets $j \geq \bar{i}$, the expected return is strictly lower in high disagreement states than in low disagreement states. □

A.5. Proof of Proposition 1

Proof. Part (i) is a direct consequence of the formula for expected excess returns in Theorem 2. For (ii), we do a Taylor expansion around $\rho = 1$ for $\Gamma^\star$: $\Gamma^\star \approx \frac{1}{r}\sum_{j \geq \bar{i}} \frac{\pi^j}{N} > 0$, so that in the vicinity of $\rho = 1$ and for $j \geq \bar{i}$,

$$\mathbb{E}[R^j(\lambda)] \approx \frac{1}{\gamma}\left(b_j \sigma_z^2 + \frac{\sigma_\epsilon^2}{N}\right) - \frac{1+r-(2\rho-1)}{(1+r) - (2\rho-1) + \frac{\theta\rho(1-\rho)}{\gamma}\Gamma^\star}\pi^j$$

The slope of the security market line for assets $i < \bar{i}$ (expressed as a function of b_i – it would be equivalent as a function of β_i) is thus strictly lower for $i < \bar{i}$ than for $i \geq \bar{i}$ in the vicinity of $\rho = 1$, which proves (ii). (iii) can also be seen directly from the previous Taylor expansion and making λ grows to infinity. (iv) is also a direct consequence of the formula for expected excess returns in Theorem 2. □