An Analytical Model for Sequential Investment Opportunities

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Abstract

We provide an analytical solution for American perpetual compound options, that do not rely on a bivariate or multivariate distribution function. This model is especially applicable for a real sequential investment opportunity, such as a series of drug development, tests and clinical trials, where the project can be cancelled at any time, and where the probability of failure declines over stages of completion. The effect of changing input parameter values can clearly be seen in terms of resulting overall project process volatility, and the effective mark-up factor which justifies continuing with each investment stage. In the base case, the effective markup factor increases as the stage nears completion if the project failure declines, although the absolute threshold of the project value less the remaining stage investment costs declines. This is consistent with the effect of decreases in project value volatility. Other results are not always intuitive, with different signed vegas and chi's for different investment stages and degrees of moneyness. This study appears to be a unique approach, which yields the threshold project value relative to investment costs that justifies investment at each stage, with no timing restrictions.

1 Introduction

We extend the result of Tourinho (1979) for a single investment opportunity to the case of a project comprising a multiple sequential investment opportunity, while retaining the simplicity of a closed-form solution. Our solution depends on assuming a probability of catastrophic failure at each investment stage that declines in value as the project nears completion, which is a characteristic of many R&D, exploration and infrastructure projects

We conceive a real sequential investment opportunity as a set of distinct, ordered investments that have to be made before the project can be completed. The project can then be interpreted as a collection of investment stages, such that no stage investment, except the first, can be started until the preceding stage has been completed. Success at each stage is not guaranteed because of the possibility of a catastrophic failure that reduces the option value to zero. The project value is realized when all the stages have been successfully completed. The following four-stage opportunity provides an illustration: (i) undertaking basic research. (ii) developing a marketable product, (iii) testing its viability and (iv) implementing the infrastructure for launch and delivery. Bearing in mind that a project can be composed of any number of distinct stages, multiple sequential investment opportunities are common amongst industries as diverse as oil exploration and mining, aircraft manufacture, pharmaceuticals and consumer electronics.

Schwartz and Moon (2000) illustrate a new drug development process which consists of four distinct phases, each with a positive probability of failure, although not necessarily declining over time. Cortazar, Schwartz and Casassus (2003) describe four natural resource exploration stages of a project with technical success probability increasing over each phase, and then a production phase which is subject to commodity price uncertainty. Pennings and Sereno (2011) describe a typical development path of a new medicine over seven phases, with a probability of failure declining over time.

Making an investment at a stage depends on whether the prevailing project value is of sufficient magnitude to economically justify committing the investment cost, or whether it is more desirable to wait for more favorable conditions. In our formulation, these conditions are represented by the prevailing project value and investment cost, which are both treated as stochastic, and possibly correlated. After making the stage investment, there is no absolute guarantee that the stage will be successfully completed, because of the presence of irresolvable difficulties in converting intentions into reality owing to technological, technical or market impediments. This means that the stage investment opportunity is subject to a catastrophic failure that causes the option value to be entirely destroyed, and the project as an entity becomes irredeemably lost. Our aim is to analyze this sequential investment opportunity under the three sources of uncertainty, the stochastic project value and the investment cost, and the probability of a catastrophic failure, so to be able to produce a closed-form rule on the investment decision at each of the project stages.

Although the single-stage investment opportunity model of Tourinho (1979), or McDonald and Siegel (1986) yields a closed-form solution, this degree of analytical elegance has not been achieved for the multi-stage sequential investment opportunity. Dixit and Pindyck (1994) identify the rule for a two-stage sequential investment but for fixed investment costs. Their solution, based on American perpetuity options, is identical to the one-stage model but with accumulated costs. Nevertheless, it is important to solve the sequential investment problem because amongst other things, the project value may vary between succeeding stages and the option value at each stage needs to be evaluated. Their resolution is an appeal to the time-to-build model of Majd and Pindyck (1987). In this representation, firms can invest continuously, at a rate no greater than a specified maximum, until the project has been completed, but investment may be temporarily halted at any time and subsequently re-started, albeit at a zero cost. The solution, evaluated by using numerical methods, shows the importance of the project value volatility in deciding whether or not to suspend investment activities. Even though the investment levels can be managed, it is essentially a single stage representation. Schwartz and Moon (2000) extend the model by including the possibility of a catastrophic failure and the presence of multiple stages, but their solution again rests on numerical methods.

Other authors simplify the multiple investment stage problems for obtaining a meaningful solution. By assuming a fixed time between stages, Bar-Ilan and Strange (1998) formulate a two-stage sequential investment model and obtain a solution by treating the option as European. Building on the valuation of sequential exchange opportunities by Carr (1988), Lee and Paxson (2001) use an element of European style compound options (and approximation of an American option phase) for formulating a two-stage sequential investment. Brach and Paxson (2001) examine a two-stage sequential investment opportunity similar to the formulation currently under study but they confine their attention more to valuation. Childs and Triantis (1999) formulate a multiple sequential investment model with interaction and obtain a solution through using a trinomial lattice. For all of these expositions, the solution is either not analytical or is restricted to only two stages.

Agliardi and Agliardi (2003, 2005), Andergassen and Sereno (2012), Gukhal (2004), Huang and Pi (2009), Lee, Yeh and Chen (2008), Pendharkar (2010) and Pennings and Sereno (2012) and other authors study N phases for a sequential option, often with a geometric Brownian motion combined with downward jumps, but typically the options are European, so the optimal timing for the investment is not computed. Cassimon et al. (2004), Cassimon et al. (2011) and Cortelezzi and Villani (2009) study American-type investment options, but provide either a Monte Carlo solution or a solution based on the complex multivariate distribution available in some mathematical programmes.

The aim of this paper is to revisit the sequential investment model originally specified by Dixit and Pindyck (1994). Combinations of three distinct sources of uncertainty associated with project value, investment cost and catastrophic failure are proposed as possible contenders for reaching a meaningful solution. Amongst these, we find that the uncertainty, at each stage, regarding the possibility of a catastrophic failure that causes the "sudden death" for the project is crucial. Although the uncertain project value is normally an essential ingredient of the real option model, it alone cannot yield a meaningful solution as established by Dixit and Pindyck (1994). However, a meaningful solution does arise when the sequential investment opportunity is considered in conjunction with the failure probability. The presence of an uncertain investment cost is not critical to obtaining a meaningful solution, but its inclusion does create a richer representation.

The major analytical findings for the sequential investment model are developed in Section 2. Based on the three sources of uncertainty, the model is presented first for a one-stage opportunity, and then incrementally developed for a two-, three- and finally (in Appendix C) for an N -stage sequential investment opportunity. We develop closed-form solutions for whether or not to commit investment at a particular stage and for the real option value at each stage. In Section 3, we obtain further insights into the model behavior through numerical illustrations. The last section summarizes some advantages and limitations of our model and suggests plausible extensions.

2 Sequential Investment Model

A firm, which can be treated as being a monopolist in its market, is considering an investment project made up of a discrete number of sequential stages, each involving an individual non-zero investment cost. The project as an entity is not fully implemented and the project value not realized until all of the sequential stages have been successfully completed. Each successive investment stage relies on the successful completion of the investment made at the preceding stage, but the stage timing is not specified. We order each investment stage by the number J of remaining stages, including the current one, until project completion. Although it may be more natural to label the initial stage of the project as 1, a reverse ordering is used since a backwardation process is used in deriving the solution. First, we examine the decision making position for the ultimate stage where $J=1$, and then by replication for the preceding stages, incrementally. At the ultimate stage, the firm is considering the decision whether or not to make an investment in a real asset. This is decided by whether or not the option value at $J=1$ fully compensates the expected net present value of the cash flow stream rendered by the asset. At the penultimate stage $J=2$, the firm is considering whether to make an expenditure to obtain the investment option at $J=1$. This decision rests on whether or not the option value at $J=2$ fully compensates the net option value at $J=1$. This procedure is then replicated incrementally for stages greater than 2. If the completion of any stage J occurs at time T_J, then $T_J > T_{J+1}$ for all positive integers J since the stages have to be completed consecutively.

A representation of the sequential investments process for a $J = N$ stage project is illustrated in Figure 1. This figure reveals the ordered sequence of stage investments comprising the project. It also shows that after an investment, the possible outcomes are success and failure. If all the stage outcomes are successful, then the entire project is successfully completed and its value can be realized. However, there is a possibility of failure at each stage. Although the investment is committed, the stage may not be successfully completed owing to fundamental irresolvable technical or market impediments, in which case, the option value instantly falls to zero and the project is abandoned without any value. The probability of failure at stage J is denoted by λ_J where $0 \le \lambda_J < 1 \, \forall J$. Situations do arise when an investment can produce an innovative breakthrough and generate an unanticipated increase in the project value, but we have ignored this possibility. Also, other forms of optionality, such as terminating a project before completion for its abandonment value, are not considered.

---- *Figure 1 about here* ----

The value of the project is defined by V. The investment expenditure made at any stage J is denoted by K_J for all possible values of J. Both the project value and the set of investment expenditures are treated as stochastic. It is assumed that they are individually well described by the geometric Brownian motion process[1]:

$$\mathrm{d}\, X = \alpha_X X \mathrm{d}\, t + \sigma_X X \mathrm{d}\, z_X \,, \tag{1.1}$$

for $X \in \{V, K_J \; \forall \, J\}$, where α_X represent the respective drift parameters, σ_X the respective instantaneous volatility parameter, and $\mathrm{d}\, z_X$ the respective increment of a standard Wiener process. Dependence between any two of the factors is represented by the covariance term; so,

[1] Many authors assume a mixed jump diffusion process for the underlying values, but in this case the entire project fails, perhaps due to a collapse in the project value, or escalation of the investment cost, or other reasons, so the jump process is not confined to a particular element.

for example, the covariance between the real asset value and the investment expenditure at stage J is specified by:

$$\mathrm{Cov}\left[\mathrm{d}V, \mathrm{d}K_J\right] = \rho_{VK_J}\sigma_V\sigma_{K_J}\,\mathrm{d}t.$$

Different stages may have different factor volatilities and correlations. The risk-free rate is r, and the investment expenditure at each stage K is assumed to be instantaneous.

2.1 One-Stage Model

The stage $J = 1$ model represents the investment opportunity for developing a project value V following the investment cost K_1, given that the research effort may fail totally with probability λ_1. We only provide here the main results since the solution is directly obtainable from McDonald and Siegel (1986). An alternative solution developed by Adkins and Paxson (2011) and applied to the one-stage model is briefly described in Appendix A, since it naturally extends to dimensions greater than two[2].

The value F_1 of the investment opportunity at stage $J = 1$ depends on the project value and the investment cost, so $F_1 = F_1(V, K_1)$. By Ito's lemma, the risk neutral valuation relationship is:

$$\begin{aligned}\tfrac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F_1}{\partial V^2} + \tfrac{1}{2}\sigma_1^2 K^2 \frac{\partial^2 F_1}{\partial K_1^2} + \rho_{VK_1}\sigma_V\sigma_{K_1} VK_1 \frac{\partial^2 F_1}{\partial V \partial K_1} \\ + \theta_V V \frac{\partial F_1}{\partial V} + \theta_{K_1} K_1 \frac{\partial F_1}{\partial K_1} - (r + \lambda_1) F_1 = 0,\end{aligned} \tag{1.2}$$

where the θ_X for $X \in \{V, K_J\ \forall J\}$ denote the respective risk neutral drift rate parameters. The generic solution to (1.2) is the two-factor power function:

$$F_1 = A_1 V^{\beta_1} K_1^{\eta_{11}}, \tag{1.3}$$

[2] The additional subscripts indicating the relevant quadrant are explained in Appendices A and B.

where β_1 and η_{11} denote the generic unknown parameters for the two factors, project value and investment cost, and A_1 denotes a generic unknown coefficient. In this notation, the first subscript for A_1, β_1 and η_{11} refers to the specific stage under consideration, while the second subscript of η_{10} refers to any feasible successive stage, which only becomes relevant for $J > 1$. By substituting (1.3) in (1.2), the power function satisfies the valuation relation with characteristic root function:

$$\begin{aligned} &Q_1(\beta_1, \eta_{11}) \\ &= \tfrac{1}{2}\sigma_V^2 \beta_1 (\beta_1 - 1) + \tfrac{1}{2}\sigma_{K_1}^2 \eta_{11}(\eta_{11} - 1) + \rho_{VK_1}\sigma_V \sigma_{K_1}\beta_1\eta_{11} + \theta_V\beta_1 + \theta_{K_1}\eta_{11} - (r + \lambda_1) = 0. \end{aligned} \tag{1.4}$$

Since a justified economic incentive to exercise the stage-one option exists provided that the project value is sufficiently high and the investment cost is sufficiently low, and the incentive intensifies for project value increases and investment cost decreases, we conjecture that $\beta_1 = \beta_{12} \geq 0$ and $\eta_{11} = \eta_{112} < 0$. Also $A_1 = A_{12} > 0$ since the option value is positive. Then (1.3) becomes:

$$F_1 = A_{12} V^{\beta_{12}} K_1^{\eta_{112}}. \tag{1.5}$$

The threshold levels for the project value and the investment cost signaling the optimal exercise for the investment option at stage $J = 1$ are denoted by $\hat{V}_1$ and $\hat{K}_{11}$, respectively. The value matching relationship describes the conservation equality at optimality that the option value $\hat{F}_1 = F_1(\hat{V}_1, \hat{K}_{11})$ exactly compensates the net asset value $\hat{V}_1 - \hat{K}_{11}$. Then:

$$A_{12} \hat{V}_1^{\beta_{12}} \hat{K}_{11}^{\eta_{112}} = \hat{V}_1 - \hat{K}_{11}. \tag{1.6}$$

The first order condition for optimality is characterized by the two associated smooth pasting conditions, one for each factor, Samuelson (1965) and Dixit (1993). These can be expressed as:

$$A_{12} \hat{V}_1^{\beta_{12}} \hat{K}_{11}^{\eta_{112}} = \frac{\hat{V}_1}{\beta_{12}} = -\frac{\hat{K}_{11}}{\eta_{102}}. \tag{1.7}$$

Since the option value is always non-negative, $A_{12} \geq 0$. Also, (1.7) corroborates our conjecture that $\beta_{12} \geq 0$ and $\eta_{112} < 0$. Together, (1.6) and (1.7) demonstrate Euler's result on homogeneity degree-one functions, Sydsæter et al. (2005), so $\beta_{12} + \eta_{112} = 1$. Replacing η_{112} by $1 - \beta_{12}$ in (1.4) yields:

$$Q_1\left(\beta_{12}, 1-\beta_{12}\right) = \tfrac{1}{2}\sigma_1^2\beta_{12}\left(\beta_{12}-1\right) + \beta_{12}\left(\theta_V - \theta_{K_1}\right) - \left(r + \lambda_1 - \theta_{K_1}\right) = 0, \tag{1.8}$$

where $\sigma_1^2 = \sigma_V^2 + \sigma_{K_1}^2 - 2\rho_{V,K_1}\sigma_V\sigma_{K_1}$. From (1.8), $\beta_{12} = \phi_1$ is the positive root solution for a quadratic equation, which is greater than 1. Further, the threshold levels are related by:

$$\hat{V}_1 = \frac{\beta_{12}}{\beta_{12}-1}\hat{K}_{11}, \tag{1.9}$$

with $A_{12} = \beta_{12}^{-\beta_{12}}\left(\beta_{12}-1\right)^{\beta_{12}-1}$. The markup factor is simply $\dfrac{\beta_{12}}{\beta_{12}-1} = \hat{V}_1 / \hat{K}_{11}$.

Finally, the option threshold value at the $J = 1$ stage defined by $\hat{F}_1 = F_1\left(\hat{V}_1, \hat{K}_{11}\right)$ is:

$$\hat{F}_1 = \frac{\hat{V}_1}{\beta_{12}}. \tag{1.10}$$

Applying Ito's lemma to (1.5):

$$\mathrm{d}F_1 = \theta_{F_1} F_1 \mathrm{d}t + \sigma_{F_1} F_1 \mathrm{d}z_{F_1}, \tag{1.11}$$

where

$$\theta_{F_1} = \tfrac{1}{2}\beta_{12}\left(\beta_{12}-1\right)\left\{\sigma_V^2 + \sigma_{K_1}^2 - 2\rho_{VK_1}\sigma_V\sigma_{K_1}\right\} + \beta_{12}\theta_V + \left(1-\beta_{12}\right)\theta_{K_1},$$

$$\sigma_{F_1}^2 = \beta_{12}^2\sigma_V^2 + \left(1-\beta_{12}\right)^2\sigma_{K_1}^2 + 2\beta_{12}\left(1-\beta_{12}\right)\rho_{VK_1}\sigma_V\sigma_{K_1}.$$

Under risk neutrality, the expected return on the option equals the risk-free rate adjusted by the probability of failure, so $\theta_{F_1} = r + \lambda_1$, which is borne out by Q_1, (1.8).

2.2 Two-Stage Model

At the preceding stage, $J = 2$, the firm examines the viability of committing an investment K_2 to acquire the option to invest F_1 by comparing the value of the compound option F_2 with the net benefits $F_1 - K_2$. Because of (1.3), F_2 depends on the three factors V, K_1 and K_2, so $F_2 = F_2\left(V, K_1, K_2\right)$. By Ito's lemma, the risk neutral valuation relationship for F_2 is:

$$\begin{aligned}
&\tfrac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F_2}{\partial V^2} + \tfrac{1}{2}\sigma_{K_1}^2 K_1^2 \frac{\partial^2 F_2}{\partial K_1^2} + \tfrac{1}{2}\sigma_{K_2}^2 K_2^2 \frac{\partial^2 F_2}{\partial K_2^2} \\
&+\rho_{V,K_1}\sigma_V\sigma_{K_1} V K_1 \frac{\partial^2 F_2}{\partial V \partial K_1} + \rho_{V,K_2}\sigma_V\sigma_{K_2} V K_2 \frac{\partial^2 F_2}{\partial V \partial K_2} + \rho_{K_1,K_2}\sigma_{K_1}\sigma_{K_2} K_1 K_2 \frac{\partial^2 F_2}{\partial K_1 \partial K_2} \\
&\qquad + \theta_V V \frac{\partial F_2}{\partial V} + \theta_{K_2} K_2 \frac{\partial F_2}{\partial K_2} + \theta_{K_1} K_1 \frac{\partial F_2}{\partial K_1} - \left(r + \lambda_2\right) F_2 = 0.
\end{aligned} \tag{1.12}$$

We conjecture that the solution to (1.12) is a product power function, with generic form:

$$F_2 = A_2 V^{\beta_{24}} K_1^{\eta_{21}} K_2^{\eta_{22}}, \tag{1.13}$$

where β_2, η_{21} and η_{22} denote the generic unknown parameters for the three factors, project value and investment expenditure at stage-one and -two respectively, and A_2 denotes an unknown coefficient. Substitution reveals that (1.13) satisfies (1.12), with characteristic root equation:

$$\begin{aligned}
&Q_2\left(\beta_2, \eta_{21}, \eta_{22}\right) \\
&\quad = \tfrac{1}{2}\sigma_V^2 \beta_2\left(\beta_2 - 1\right) + \tfrac{1}{2}\sigma_{K_1}^2 \eta_{21}\left(\eta_{21} - 1\right) + \tfrac{1}{2}\sigma_{K_2}^2 \eta_{22}\left(\eta_{22} - 1\right) \\
&\qquad + \rho_{VK_1}\sigma_V\sigma_{K_1}\beta_2\eta_{21} + \rho_{VK_2}\sigma_V\sigma_{K_2}\beta_2\eta_{22} + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\eta_{21}\eta_{22} \\
&\qquad\quad + \theta_V\beta_2 + \theta_{K_1}\eta_{21} + \theta_{K_2}\eta_{22} - \left(r + \lambda_2\right) = 0.
\end{aligned} \tag{1.14}$$

Since the stage-two option value increases for positive changes in V_2 but for negative changes in K_{12} and K_{22}, we conjecture in Appendix B that the relevant hyper-quadrant is labeled IV where $\beta_2 = \beta_{24} \geq 0$, $\eta_{21} = \eta_{214} < 0$ and $\eta_{22} = \eta_{224} < 0$. From (1.13), the option valuation function becomes:

$$F_2 = A_{24} V^{\beta_{24}} K_1^{\eta_{214}} K_2^{\eta_{224}} . \tag{1.15}$$

We specify that the stage-two threshold levels signaling an optimal exercise are represented by $\hat{V}_2$, $\hat{K}_{21}$ and $\hat{K}_{22}$ for V, K_1 and K_2, respectively. The set $\left\{\hat{V}_2, \hat{K}_{21}, \hat{K}_{22}\right\}$ forms the boundary that discriminates between the "exercise" decision and the "wait" decision. This boundary is determined from establishing the relationship amongst $\hat{V}_2$, $\hat{K}_{21}$ and $\hat{K}_{22}$, or alternatively, from identifying the dependence of $\hat{V}_2$ with respect to $\hat{K}_{21}$ and $\hat{K}_{22}$. A stage-two option exercise occurs for the balance between the stage-two option value $A_{24} \hat{V}_2^{\beta_{24}} \hat{K}_{21}^{\eta_{214}} \hat{K}_{22}^{\eta_{224}}$ and the stage-one option value $A_{12} \hat{V}_1^{\beta_{12}} \hat{K}_{11}^{\eta_{11}}$ less the investment cost $\hat{K}_{22}$ incurred in its acquisition. This equilibrium amongst the threshold levels is the value matching relation that is expressed as:

$$A_{24} \hat{V}_2^{\beta_{24}} \hat{K}_{12}^{\eta_{214}} \hat{K}_{22}^{\eta_{224}} = A_{12} \hat{V}_2^{\beta_{12}} \hat{K}_{12}^{1-\beta_{12}} - \hat{K}_{22}, \tag{1.16}$$

where A_{12} and β_{12} are known from the evaluation for stage-one. The three smooth pasting conditions associated with (1.16), one for each of the three factors V, K_1 and K_2, respectively, can be expressed as:

$$\beta_{24} A_{24} \hat{V}_2^{\beta_{24}} \hat{K}_{12}^{\eta_{214}} \hat{K}_{22}^{\eta_{224}} = \beta_{12} A_{12} \hat{V}_2^{\beta_{12}} \hat{K}_{12}^{1-\beta_{12}}, \tag{1.17}$$

$$\eta_{214} A_{24} \hat{V}_2^{\beta_{24}} \hat{K}_{12}^{\eta_{214}} \hat{K}_{22}^{\eta_{224}} = \left(1 - \beta_{12}\right) A_{12} \hat{V}_2^{\beta_{12}} \hat{K}_{12}^{1-\beta_{12}}, \tag{1.18}$$

$$\eta_{224} A_{24} \hat{V}_2^{\beta_{24}} \hat{K}_{12}^{\eta_{214}} \hat{K}_{22}^{\eta_{224}} = -\hat{K}_{22}. \tag{1.19}$$

Since an option value is non-negative, then $A_{24} \geq 0$. This implies that $\beta_{24} \geq 0$ from (1.17), $\eta_{214} < 0$ from (1.18), and $\eta_{224} < 0$ from (1.19), which justifies our conjecture on the signs of the power parameters. Moreover, the dependence amongst the parameters can be found from combining the smooth pasting conditions and the value matching relationship. First, the comparison of (1.17) and (1.19) with (1.16) yields:

$$\eta_{224} = 1 - \frac{\beta_{24}}{\beta_{12}}, \tag{1.20}$$

which implies that $\beta_{24} > \beta_{12}$. Second, the comparison of (1.19) with (1.20) yields:

$$\eta_{214} = \frac{1-\beta_{12}}{\beta_{12}}\beta_{24}. \tag{1.21}$$

Third, a comparison of (1.20) with (1.21) yields:

$$\beta_{24} + \eta_{214} + \eta_{224} = 1. \tag{1.22}$$

The pattern amongst the parameters is highly significant. First, it leads to a simplification in calculating their solution values. If we specify $\phi_{24} = \beta_{24} / \beta_{12} \geq 0$, then by using the substitutions $\beta_{24} = \phi_{24}\beta_{12}$, $\eta_{214} = (1-\beta_{12})\phi_{24}$ and $\eta_{224} = 1-\phi_{24}$, the quadratic function Q_2 (1.14) can be expressed as:

$$\begin{aligned} &Q_2\left(\beta_{12}\phi_{24}, (1-\beta_{12})\phi_{24}, 1-\phi_{24}\right) \\ &\quad = \tfrac{1}{2}\phi_{24}\left(\phi_{24}-1\right)\sigma_2^2 + \phi_{24}\left\{\tfrac{1}{2}\beta_{12}\left(\beta_{12}-1\right)\sigma_1^2 + \beta_{12}\left(\theta_V - \theta_{K_1}\right) + \left(\theta_{K_1} - \theta_{K_2}\right)\right\} \\ &\qquad\qquad - \left(r + \lambda_2 - \theta_{K_2}\right) = 0, \end{aligned} \tag{1.23}$$

where

$$\begin{aligned} \sigma_2^2 &= \beta_{12}^2\sigma_V^2 + \left(1-\beta_{12}\right)^2\sigma_{K_1}^2 + \sigma_{K_2}^2 \\ &\quad + 2\beta_{12}\left(1-\beta_{12}\right)\rho_{VK_1}\sigma_V\sigma_{K_1} - 2\beta_{12}\rho_{VK_2}\sigma_V\sigma_{K_2} - 2\left(1-\beta_{12}\right)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}. \end{aligned}$$

The value of ϕ_{24} is evaluated as the positive root of $Q_2 = 0$, (1.23), where β_{12} is the previously calculated stage-one solution. The values of β_{24}, η_{214} and η_{224} are then obtained from ϕ_{24} and β_{12}. Subsequently, we show that ϕ_{24} is greater than 1, so $\beta_{24} > \beta_{12}$.

The second significant feature is the ease in deriving the solution. Although the solution for A_{24} and $\hat{V}_2$ as a function of $\hat{K}_{21}$ and $\hat{K}_{22}$ can be derived from the value matching relationship and the smooth pasting conditions, (1.16) - (1.19), a more convenient way is based on the homogeneity degree-one property for F_2, since the result is easily extendable for deriving the stage-three solution and beyond. The valuation function F_2 (1.13) can be expressed in the form:

$$F_2 = F_{22}\left(F_1, K_2\right) = B_2\left[F_1\left(V, K_1\right)\right]^{\phi_2} K_2^{1-\phi_2} = B_2\left[A_1 V^{\beta_1} K_1^{\eta_{11}}\right]^{\phi_2} K_2^{1-\phi_2}, \tag{1.24}$$

where $B_2 = A_2 A_1^{-\phi_2}$. In this formulation, the two-stage option value F_2 is a function of two stochastic factors: (i) the stage-one option value F_1, and (ii) the stage-two investment cost K_2. Moreover, since F_2 is characterized as homogenous degree-one and its form (1.24) exactly mirrors the stage-one investment option value F_1, the solution is directly obtainable from the results for the one-stage model. If the stage-two thresholds for optimal exercise occur at the levels $\hat{F}_{12} = F_1\left(\hat{V}_2, \hat{K}_{12}\right)$ and $\hat{K}_{22}$, for the stage-one option and the stage-two investment cost, respectively, then the stage-two value matching relationship (1.16) can be expressed as:

$$B_{24}\hat{F}_{12}^{\phi_{24}}\hat{K}_{22}^{1-\phi_{24}} = \hat{F}_{12} - \hat{K}_{22}. \tag{1.25}$$

Except for the change in variable, (1.25) is identical in form to (1.6), so $B_{24} = \left(\phi_{24} - 1\right)^{(\phi_{24}-1)} / \phi_{24}^{\phi_{24}}$, which implies:

$$A_{24} = \frac{\left(\phi_{24} - 1\right)^{(\phi_{24}-1)}}{\phi_{24}^{\phi_{24}}}\left[\frac{\left(\beta_{12} - 1\right)^{\beta_{12}-1}}{\beta_{12}^{\beta_{12}}}\right]^{\phi_{24}}, \tag{1.26}$$

so the two-stage option value is defined by:

$$F_2\left(V_2, K_1, K_2\right) = \frac{\left(\phi_{24}-1\right)^{\left(\phi_{24}-1\right)}}{\phi_{24}^{\phi_{24}}}\left[\frac{\left(\beta_{12}-1\right)^{\beta_{12}-1}}{\beta_{12}^{\beta_{12}}}\right]^{\phi_{24}} V_2^{\beta_{12}\phi_{24}} K_1^{\left(1-\beta_{12}\right)\phi_{24}} K_2^{1-\phi_{24}}. \tag{1.27}$$

Also:

$$\hat{F}_{12} = F_1\left(\hat{V}_2, \hat{K}_{12}\right) = A_{12}\hat{V}_2^{\beta_{12}} K_{12}^{1-\beta_{12}} = \frac{\phi_{24}}{\phi_{24}-1}\hat{K}_{22}, \tag{1.28}$$

so:

$$\begin{aligned}\hat{V}_2 &= \frac{\beta_{12}}{\beta_{12}-1}\left\{\frac{\phi_{24}\left(\beta_{12}-1\right)}{\phi_{24}-1}\right\}^{\frac{1}{\beta_{12}}} \hat{K}_{12}^{\frac{\beta_{12}-1}{\beta_{12}}} \hat{K}_{22}^{\frac{1}{\beta_{12}}} \\ &= \frac{\beta_{12}}{\beta_{12}-1}\left\{\frac{\beta_{24}\left(\beta_{12}-1\right)}{\beta_{24}-\beta_{12}}\right\}^{\frac{1}{\beta_{12}}} \hat{K}_{12}^{\frac{\beta_{12}-1}{\beta_{12}}} \hat{K}_{22}^{\frac{1}{\beta_{12}}}.\end{aligned} \tag{1.29}$$

For an economically meaningful solution to emerge, then from (1.28) ϕ_{24} has to exceed one. In (1.29), the threshold level for the stage-two project value $\hat{V}_2$ is related to the stage-one and stage-two investment cost levels, $\hat{K}_{12}$ and $\hat{K}_{22}$, respectively, and this relationship defines two-stage compound option and extends the single stage standard result of McDonald and Siegel (1986). The two investment cost threshold levels enter the formulation as a weighted geometric average with weights dependent on only the stage-one parameter. If the levels are specified to be equal, then the stage-two project value level $\hat{V}_2$ and the equal investment cost level are linearly related, just as for $\hat{V}_1$ and $\hat{K}_1$ at stage-one. The composite stage-two basic mark-up factor:

$$\frac{\beta_{12}}{\beta_{12}-1}\left\{\frac{\phi_{24}\left(\beta_{12}-1\right)}{\phi_{24}-1}\right\}^{\frac{1}{\beta_{12}}}$$

is composed of two components: the stage-one mark-up factor adjusted by a term reflecting the impact of the second stage. Since $\beta_{12}, \phi_{24} > 1$, the adjusting component $\left\{\phi_{24}\left(\beta_{12}-1\right)/\left(\phi_{24}-1\right)\right\}^{\frac{1}{\beta_{12}}}$ is greater than one provided $\beta_{12} > 2 - 1/\phi_{24}$, which is always true for $\beta_{12} > \phi_{24}$. However, now this basic factor is applied to the K powers $\hat{K}_{12}^{\frac{\beta_{12}-1}{\beta_{12}}}\hat{K}_{22}^{\frac{1}{\beta_{12}}}$. So for consistency with the Stage 1 markup factor, we denote the Stage 2 markup effective factor, MEF, as simply $\hat{V}_2/(\hat{K}_{12}+\hat{K}_{22})$.

Standard real-option theory tells us that the underlying volatility has a profound effect on the solution, McDonald and Siegel (1986), Dixit and Pindyck (1994). For a given value of the stage-one power parameter β_{12}, a positive change in σ_2 produces a decrease in the parameter ϕ_{24}, but an increase in $\phi_{24}/(\phi_{24}-1)$ and in the adjusting component that yields an increase in the stage-two mark-up factor. Now, the variance term σ_2 depends on the parameter β_{12} as well as the volatilities for V, K_1 and K_2, and their covariances. We first consider the consequences if all the covariances can be assumed to be zero. High values for β_{12}, which are caused by low σ_V and σ_{K_1}, tend to ratchet up the value of σ_2, while a value of β_{12} closer to 1 due to high σ_V or σ_{K_1}, tends to diminish the effect of σ_{K_1} in explaining σ_2. The importance of σ_{K_1} in determining σ_2 depends on its magnitude relative to σ_V. Further, since the value of β_{12} depends positively on the probability of a catastrophic failure at the $J=1$ stage, the importance of σ_{K_2} relative to σ_V and σ_{K_1} in explaining σ_2 diminishes as the failure probability increases. It is through this mechanism that the probabilities of catastrophic failures at succeeding stages are translated into the investment strategy at stage-two.

We now turn our attention to the effects of the covariance terms on σ_2. If $\rho_{VK_1}>0$, $\rho_{VK_2}>0$, or $\rho_{K_1K_2}<0$, then the value of σ_2 declines while the value of β_{12} increases relative to the instance of zero correlations. This can be explained in the following way. A long investment cost acts as a partial hedge for a long project value whenever $\rho_{VK_1}>0$ and $\rho_{VK_2}>0$, since a random positive (negative) movement in the investment cost is partly compensated by a movement in the same direction in the project value. (A long/short position in the investment cost might be established through fixed-price/cost-plus construction contracts). This partial hedge reduces the riskiness of the combined position, which is reflected in a lower value of σ_2. In contrast, a long investment cost and V position becomes more risky whenever ρ_{VK_1} or ρ_{VK_2} is negative. If $\rho_{K_1K_2} < 0$, then a random movement in the $J = 2$ stage investment cost tends to be followed by a movement in the opposite direction in the $J = 1$ stage investment cost, and a long $J = 2$ stage investment cost acts as a hedge against a short $J = 1$ stage investment cost. A positive movement in the $J = 2$ stage investment cost that is followed by a negative movement in the $J = 1$ stage investment cost can be interpreted as dynamic learning, since a higher than anticipated preliminary investment cost leads to a lower investment cost at a subsequent stage, while a negative movement in the $J = 2$ stage investment cost that is followed by a positive movement in the $J = 1$ stage investment cost can be interpreted as compensatory. Under-investment is corrected by over-investment at a subsequent stage. In contrast, when $\rho_{K_1K_2} > 0$, the volatility σ_2 is inflated. This can arise from a positive movement in the $J = 2$ stage investment cost that is followed by a positive movement in the $J = 1$ stage investment cost, which suggests that errors at the earlier $J = 2$ stage are compounded at the later $J = 1$ stage. However, a positive value for $\rho_{K_1K_2}$ can just as well be due to a negative movement in the $J = 2$ stage investment cost followed by a negative movement in the $J = 1$ stage investment cost. This may also represent bad news if low investment levels presage low project values. Clearly, the sensitivity of the volatility σ_2 depends on the magnitudes of the contributory quantities as well as their interactions.

By combining (1.23) with (1.8) in order to eliminate β_{12}, the Q_2 function can be expressed as:

$$Q_2 = \tfrac{1}{2}\phi_{24}\left(\phi_{24} - 1\right)\sigma_2^2 + \phi_{24}\left(r + \lambda_1 - \theta_{K_2}\right) - \left(r + \lambda_2 - \theta_{K_2}\right) = 0. \tag{1.30}$$

The parameter ϕ_{24}, which is required to be greater than one, is evaluated as the positive root of the quadratic function Q_2 (1.30). Given that $\sigma_2^2 > 0$, since it is a variance expression, then $\phi_{24} > 1$ provided that the value of Q_2 evaluated at $\phi_{24} = 1$ is negative, Dixit and Pindyck (1994). It can be observed from (1.30) that for $Q_2 < 0$ at $\phi_{24} = 1$, then $\lambda_2 > \lambda_1$, see also Figure 2.

---- *Figure 2 about here* ---

The parameter λ measures the conditional probability of a catastrophic failure at a particular stage. The existence of a solution to the sequential investment model represented by an American perpetual compound option depends crucially on the probabilities at the two stages following a distinct pattern. Although it plays an important role in deciding an acceptable investment level at each stage, the stochastic nature of the investment expenditures is not critical. The condition $\lambda_2 > \lambda_1$ for obtaining a meaningful solution continues to hold even if both σ_{K_1} and σ_{K_2} are zero, so our findings also apply for a deterministic investment cost. The only requirement for a meaningful solution to exist is that the conditional probability of a failure at the $J=2$ stage has to exceed that for the $J=1$ stage. This condition can be seen simply as a stipulation imposed by the model structure. Since $\lambda_2 > \left(1-\lambda_2\right)\lambda_1$, the failure probability at the $J=2$ stage is always greater than that for the $J=1$ stage. Alternatively, this condition could be interpreted as the presence of dynamic learning. Because of the reduction in the failure probabilities, the effect of making an investment at the $J=2$ stage is to increase the affordable amount of investment expenditure made at the subsequent stage. Ceteris paribus, project viability is able to support a higher level of investment expenditure at the next stage, and this implies some element of learning.

By applying Ito's lemma to (1.24), then the return on the option F_2 is:

$$\frac{\mathrm{d}F_2}{F_2} = \theta_{F_2}\mathrm{d}t + \sigma_{F_2}\mathrm{d}z_{F_2}\,, \tag{1.31}$$

where:

$$\theta_{F_2} = \tfrac{1}{2}\phi_{24}\left(\phi_{24}-1\right)\left\{\sigma_{F_1}^2 + \sigma_{K_2}^2 - 2\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2}\right\} + \phi_{24}\theta_{F_1} + \left(1-\phi_{24}\right)\theta_{K_2}\,,$$

$$\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2} = \beta_{12}\rho_{VK_2}\sigma_V\sigma_{K_2} + \left(1-\beta_{12}\right)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\,,$$

$$\sigma_{F_2}^2 = \phi_{24}^2\sigma_{F_1}^2 + \left(1-\phi_{24}\right)^2\sigma_{K_2}^2 + 2\phi_{24}\left(1-\phi_{24}\right)\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2}\,.$$

Under risk neutrality, $\theta_{F_2} = r + \lambda_2$, which is borne out by the Q_2 function.

2.3 Three-Stage Model

Since the extension of the sequential investment model to the $J = 3$ stage is achieved by replication, we only provide the crucial results with only a basic explanation. Then, the comparison of the results for each of the three stages facilitates the formulation of a more general result for a $J = N$ stage project.

The value of the option to invest at the $J = 3$ stage F_3 depends on the project value V, and the investment costs at the $J = 1$, $J = 2$ and $J = 3$ stages, K_1, K_2 and K_3, respectively, so $F_3 = F_3\left(V, K_1, K_2, K_3\right)$. Using Ito's lemma, it can be shown that the risk neutral valuation relationship for F_3 is a four-dimensional partial differential equation, whose solution is the product power function:

$$F_3 = A_3 V^{\beta_3} K_1^{\eta_{13}} K_2^{\eta_{23}} K_3^{\eta_{33}}, \tag{1.32}$$

with characteristic root equation:

$$\begin{aligned} & Q_3\left(\beta_3, \eta_{13}, \eta_{23}, \eta_{33}\right) \\ & = \tfrac{1}{2}\sigma_V^2 \beta_3\left(\beta_3 - 1\right) + \tfrac{1}{2}\sigma_{K_1}^2 \eta_{13}\left(\eta_{13} - 1\right) + \tfrac{1}{2}\sigma_{K_2}^2 \eta_{23}\left(\eta_{23} - 1\right) + \tfrac{1}{2}\sigma_{K_3}^2 \eta_{33}\left(\eta_{33} - 1\right) \\ & \quad + \rho_{VK_1}\sigma_V \sigma_{K_1} \beta_3 \eta_{13} + \rho_{VK_2}\sigma_V \sigma_{K_2} \beta_3 \eta_{23} + \rho_{VK_3}\sigma_V \sigma_{K_3} \beta_3 \eta_{33} \\ & \quad + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\eta_{13}\eta_{23} + \rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\eta_{13}\eta_{33} + \rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\eta_{23}\eta_{33} \\ & \quad + \theta_V \beta_3 + \theta_{K_1}\eta_{13} + \theta_{K_2}\eta_{23} + \theta_{K_3}\eta_{33} - \left(r + \lambda_3\right) = 0. \end{aligned} \tag{1.33}$$

The function Q_3 specifies a hyper-ellipse that has a presence in all possible quadrants. The relevant quadrant is where $\beta_3 > 0$, $\eta_{13} < 0$, $\eta_{23} < 0$ and $\eta_{33} < 0$, since we expect the stage-three investment option to become more valuable and its value to rise because of a project value increase but an investment cost decrease. For convenience, we suppress the subscript designating the relevant quadrant.

Alternatively, the valuation function F_3 can be expressed as:

$$F_3 = F_{33}\left(F_2, K_3\right) = B_3\left[F_2\left(V, K_1, K_2\right)\right]^{\phi_3} K_3^{1-\phi_3}, \tag{1.34}$$

where $\beta_3 = \phi_3\beta_2 = \phi_3\phi_2\phi_1$, $\eta_{13} = \phi_3\eta_{12} = \phi_3\phi_2\left(1-\phi_1\right)$, $\eta_{23} = \phi_3\eta_{22} = \phi_3\left(1-\phi_2\right)$, and $\eta_{33} = \left(1-\phi_3\right)$, with $\phi_1 = \beta_1$. The coefficient B_3 is determined as:

$$B_3 = \frac{1}{\phi_3 - 1}\frac{\left(\phi_3 - 1\right)^{\phi_3}}{\phi_3^{\phi_3}} = A_3 A_2^{-\phi_3}$$

At the stage-three investment decision, the thresholds signaling an optimal exercise for the stage-three option value F_3, the stage-two option value F_2 and the stage-three investment cost K_3 are denoted by $\hat{F}_3 = A_3\hat{V}_3^{\beta_3}\,\hat{K}_{13}^{\eta_{13}}\,\hat{K}_{23}^{\eta_{23}}\,\hat{K}_{33}^{\eta_{33}}$, $\hat{F}_{23} = F_2\left(\hat{V}_3, \hat{K}_{13}, \hat{K}_{23}\right)$ and $\hat{K}_{33}$, respectively. Value conservation at the stage-three investment holds when the stage-three option value $\hat{F}_3$ exactly compensates the stage-two option value $\hat{F}_{23}$ less the investment cost $\hat{K}_{33}$. The value matching relationship becomes:

$$\hat{F}_3 = B_3\hat{F}_{23}^{\phi_3}K_{33}^{1-\phi_3} = \hat{F}_{23} - \hat{K}_{33}. \tag{1.35}$$

The optimal stage-three investment solution is obtained from the two smooth pasting conditions associated with the value matching relationship (1.35) and can be expressed as:

$$\hat{F}_{23} = \frac{\phi_3}{\phi_3 - 1}\hat{K}_{33}. \tag{1.36}$$

Since from (1.27):

$$\hat{F}_{23} = F_2\left(\hat{V}_3, \hat{K}_{13}, \hat{K}_{23}\right) = \frac{\left(\phi_2 - 1\right)^{(\phi_2 - 1)}}{\phi_2^{\phi_2}} \left[\frac{\left(\phi_1 - 1\right)^{\phi_1 - 1}}{\phi_1^{\phi_1}}\right]^{\phi_2} \hat{V}_3^{\phi_1\phi_2} \hat{K}_{13}^{(1-\phi_1)\phi_2} \hat{K}_{23}^{1-\phi_{24}}, \tag{1.37}$$

then from (1.36):

$$\hat{V}_3 = \left\{\frac{\phi_3}{\phi_3 - 1} \frac{\phi_2^{\phi_2}}{\left(\phi_2 - 1\right)^{(\phi_2 - 1)}} \left[\frac{\phi_1^{\phi_1}}{\left(\phi_1 - 1\right)^{\phi_1 - 1}}\right]^{\phi_2}\right\}^{1/\phi_1\phi_2} \hat{K}_{13}^{(\phi_1 - 1)/\phi_1} \hat{K}_{23}^{(\phi_2 - 1)/\phi_1\phi_2} \hat{K}_{33}^{1/\phi_1\phi_2} \tag{1.38}$$

Clearly, an economically meaningful solution is only obtainable provided ϕ_3 exceeds 1, which implies that $\beta_3 > \beta_2$. If the investment cost threshold levels for the three stages are all equal, then the project value threshold is a linear relationship of this equal investment cost threshold. The stage-three mark-up factor in (1.38) can be expressed as:

$$\left\{\frac{\phi_2}{\phi_2 - 1} \frac{\phi_1^{\phi_1}}{\left(\phi_1 - 1\right)^{\phi_1 - 1}}\right\}^{1/\phi_1} \left\{\left(\phi_2 - 1\right) \frac{\phi_3}{\phi_3 - 1}\right\}^{1/\phi_1\phi_2}$$

where the first term is the stage-two mark-up factor and the second term $\left\{\left(\phi_2 - 1\right)\phi_3 \big/ \left(\phi_3 - 1\right)\right\}^{1/\phi_1\phi_2}$ denotes the adjusting component. The stage-three mark-up factor exceeds the stage-two mark-up factor provided $\left(\phi_2 - 1\right)\phi_3 \big/ \left(\phi_3 - 1\right) > 1$ or $\phi_2 > 2 - 1/\phi_3$, which is true for $\phi_2 > \phi_3$. For consistency with the Stage 1 markup factor, we denote the Stage 3 markup effective factor, MEF, as simply $\hat{V}_3 / (\hat{K}_{13} + \hat{K}_{23} + \hat{K}_{33})$.

In (1.38), the solution to the boundary discriminating between investing and not investing at stage-three requires evaluating only ϕ_3, since ϕ_2 and ϕ_1 are presumed to have been calculated at each of the subsequent two stages. By eliminating η_{13}, η_{23} and η_{33} from (1.33) yields after some simplification:

$$
\begin{aligned}
&Q_3\left(\phi_3\phi_2\phi_1,\phi_3\phi_2\left(1-\phi_1\right),\phi_3\left(1-\phi_2\right),\left(1-\phi_3\right)\right)\\
&\quad=\tfrac{1}{2}\sigma_3^2\phi_3\left(\phi_3-1\right)\\
&\qquad+\phi_3\left[\tfrac{1}{2}\sigma_2^2\phi_2\left(\phi_2-1\right)+\phi_2\left\{\tfrac{1}{2}\sigma_1^2\phi_1\left(\phi_1-1\right)+\theta_V\phi_1+\theta_{K_1}\left(1-\phi_1\right)\right\}+\theta_{K_2}\left(1-\phi_2\right)-\theta_{K_3}\right]\\
&\qquad-\left(r+\lambda_3-\theta_{K_3}\right)=0,
\end{aligned}
$$

where:

$$
\begin{aligned}
\tfrac{1}{2}\sigma_3^2&=\tfrac{1}{2}\sigma_V^2\phi_2^2\phi_1^2+\tfrac{1}{2}\sigma_{K_1}^2\phi_2^2\left(1-\phi_1\right)^2+\tfrac{1}{2}\sigma_{K_2}^2\left(1-\phi_2\right)^2+\tfrac{1}{2}\sigma_{K_3}^2\\
&\quad+\rho_{VK_1}\sigma_V\sigma_{K_1}\phi_1\left(1-\phi_1\right)\phi_2^2+\rho_{VK_2}\sigma_V\sigma_{K_2}\phi_1\phi_2\left(1-\phi_2\right)\beta_3\eta_{23}-\rho_{VK_3}\sigma_V\sigma_{K_3}\phi_1\phi_2\\
&\quad+\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\left(1-\phi_1\right)\phi_2\left(1-\phi_2\right)-\rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\left(1-\phi_1\right)\phi_2-\rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\left(1-\phi_2\right).
\end{aligned}
$$

Then after further simplification, we obtain:

$$Q_3=\tfrac{1}{2}\sigma_3^2\phi_3\left(\phi_3-1\right)+\phi_3\left(r+\lambda_2-\theta_{K_3}\right)-\left(r+\lambda_3-\theta_{K_3}\right)=0. \tag{1.39}$$

The value of ϕ_3 is the positive root solution to (1.39). By applying a similar argument as before, its value exceeds one provided that $\lambda_3>\lambda_2$. Further, it depends not only on the stage-three catastrophic failure probability λ_3 but also the probability λ_2 at the next stage, as well as on the composite variance term σ_3^2. This variance term is defined as the sum of variance and covariances amongst the factors, the project value and the stage-three, -two and -one investment costs, weighted by combinations of ϕ_3, ϕ_2 and ϕ_1. Because of this, the stage-three investment commitment is decided by the properties of both the current and subsequent stages.

2.4 N-Stage Model

The solution to the $J=N>1$ stage of the sequential investment model is derived from the results for the $J=2$ and $J=3$ stages by induction, as shown in Appendix C.

3 Numerical Illustrations

We have presented a general analytical framework for evaluating a sequential investment project that can be characterized by $J = N$ successive investment stages. This framework incorporates three sources of uncertainty: (i) uncertainty regarding the asset value on completion of the project, (ii) uncertainty regarding the cost of the investment at each of the stages, and (iii) uncertainty, at each stage, regarding the possibility of a catastrophic failure that causes the "sudden death" of the project, by imposing the stage option value to collapse to zero. Further, the framework allows for the first two types of uncertainty to co-vary. Analysis of the general framework also yields closed-form solutions for the optimal investment strategy as the maximal amount of investment allowable at each stage according to the project's prevailing value, or alternatively the minimal prevailing value that would justify proceeding with the stage investment, if the investment cost is at the assumed base value.

We establish that for an investment to be economically justified at each stage, the prevailing project value has to exceed the anticipated investment cost. Also we assumed that the probability of a catastrophic failure at each stage declines successively as the stage approaches completion.

To obtain additional insights into the behaviour of the analytical framework, we conduct some numerical evaluations on an illustration involving a 4-stage sequential investment project using the base case information exhibited in Table 1. The set of probabilities of catastrophic failure at the stages adheres to the condition $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Initially, the variances for the investment costs at the four stages have been set to be equal and the covariance terms between the five factors to equal zero.

---- *Table 1 about here* ----

First, we consider the results for the base case information, and then examine the impact of key sensitivities.

Table 2 shows the results calculated from the values exhibited in Table 1, using the backwardation principle so the $J = 1$ stage is enumerated first, then the $J = 2$ stage, and so on.

The volatilities at each of the 4 stages, σ_1, σ_2, σ_3 and σ_4, are evaluated from (1.49), the parameters ϕ_J for $J=1$ from (1.8) and for $J=2,3,4$ from (1.49), and the mark-up factors for each of the 4 stages from (1.47). Table 2 illustrates that the volatilities at each stage, σ_J for $J=1,2,3,4$, increase in magnitude as the stage in question becomes more distant from completion. This finding is in line with expectations, since the volatility depends not only on the volatilities for the project value and the current stage investment cost but also on the cascading effect of the investment cost volatilities and parameter values for all possible subsequent stages. As expected, the parameter values ϕ_J for $J=1,2,3,4$ are all greater than one. This feature arises owing to the pattern of failure probabilities specified in Table 1. Even though $\lambda_4 > \lambda_3 > \lambda_2 > \lambda_1$, as required by the model, this does not imply that the ϕ_J necessarily follow a declining pattern. The values for the ϕ_J depend not only on the volatility for the stage in question but also on the failure probability λ_J. These two effects work in opposing directions. While an increase in volatility for the J stage yields an increase in ϕ_J, an increase in the failure probability λ_J leads a decline in ϕ_J. Although the former is more dominant according (Dixit and Pindyck 1994), it is conceivable that $\phi_{J+1} > \phi_J$. Note that with these parameter values, $\hat{V}$ increases with the distance of the stage from completion, and with the stage volatility, but the excess of the $\hat{V}$ over the assumed investment cost is variable over each stage. The real option value (ROV), which is the option to continue the next stages if $V < \hat{V}$, and otherwise V less the remaining investment costs (or zero) also varies by stage. In all stages, the ROV is the option value, unless V=100 > $\hat{V}$, when the ROV is the intrinsic value. The ROV increases as V increases for all stages unless V> $\hat{V}$, at which point the investment should be made. So in the base case, in stages 1 and 2 and in all stages when V>127.2, ROV is the intrinsic value, and otherwise the (unexercised) option value. But there are some apparent anomalies such as stage 3, when the ROV is slightly less than the intrinsic value, showing that these are not ordinary options.

---- *Table 2 about here* ----

The middle column in the first panel is labeled the mark-up effective factor (MEF). This reveals the MEFs to be in excess of one, as required. The MEF is a critical element of the investment strategy, since it stipulates for each stage that for an investment at that stage to be economically justified, the ratio of the anticipated project value to investment cost has to be at least equal to the MEF. According to our results, the MEFs decrease in magnitude as the stage becomes more distant from completion, but the project value that justifies investment less the assumed remaining investment costs declines, given these parameter values.

3.1 Probability of Failure

We examine how the probability of a catastrophic failure influences the solution initially by increasing its value, λ_J for $J = 1,2,3,4$ by a constant amount from 5% to 30% (0% is the base case). The results for ROV do not necessarily conform with expectations. The increase in probability at each stage has the consequence of raising the volatility, of lowering the parameter value ϕ_J and lowering the MEF, but only for stages 2, 3 and 4. For the $J = 1$ stage, since an increase in λ_1 effectively raises the discount rate but leaves the volatility σ_1 unaffected, there is a consequential rise in the parameter value ϕ_J and fall in the MEF reflecting a greater urgency in exercising the option, Dixit and Pindyck (1994). However, this feature is not replicated for stages $J = 2,3,4$ since the effect of the increased discount rate due to the failure probability increase is dominated by the cascading impact of the failure probability at the stage $J = 1$ on the volatility σ_J at stages $J = 2,3,4$. A change in the failure probability λ_J for $J = I$ has both a direct effect on the solution at the stage $J = I$ as well as an indirect effect on the solution at the stages $J > I$ due to the cascading impact of λ_I on the volatilities.

For every stage, an increase in the probability of failure results in a decline of the stage V which would justify an investment at that stage. The ROV does change along the stages or with increases in the stage failure probabilities. Later stage 1 & 2 ROV are reduced by failure probability increases, but earlier 3 & 4 stage ROVs are initially increased by modest increases in failure probabilities. But the threshold $\hat{V}$ decreases for each stage as the probability of failure increases, a surprising result.

---- *Figure 3 about here* ----

3.2 Volatility

For the single-stage investment opportunity, an increase in project value volatility is normally accompanied with a fall in the parameter value, a rise in the MEF, a rise in $\hat{V}$ and a rise in the ROV (positive "vega"). We obtain this finding for the multi-stage sequential investment opportunity for some stages and for some ranges of volatility. Figure 4 illustrates the effect of changing the project value volatility along a range of 0% to 60%. For all stages, an increase in the value volatility results in an increase of the threshold V that justifies an investment at that stage. For the early stages, $J = 3,4$, while there is a fall in the parameter value and a rise in the MEF, there is, in contrast, a decrease in the ROV for certain volatility ranges. This is an instance of a negative "vega" for these parameter values. So high project volatility and a high probability of failure do not always increase option value in sequential investments. However, the V thresholds for all stages increase with increases of volatility, which is consistent with traditional real option theory.

A somewhat similar pattern of effects arising from the increase in project value volatility is shown for an increase in the investment cost volatility,. Figure 5 illustrates the impact of increasing all investment cost volatilities along a range of 0% to 30%. There are both positive and negative vega effects, depending on the stages and the level of volatility. However, the V thresholds for all stages increase with increases of investment cost volatility (with a base correlation of zero, showing that the effect on the threshold V is not dramatic), which is consistent with traditional real option theory.

---- *Figure 4 and Figure 5 about here* ----

3.3 Correlation

Changes in the correlation coefficients impact on the solution through the relevant stage volatility, σ_J for $J = 1,2,3,4$, which in turn influences the parameter value. Further, since the volatility at the preceding stage σ_{J+1} depends on the volatility at the current stage σ_J, changes in the correlation coefficient cascade through the volatilities of the preceding stages. Theoretically, we argue that owing to the hedging effect, a positive change in the correlation between the project value and the investment cost depresses the stage volatility, which in turn raises the parameter value, while a negative change in the correlation between value and investment cost increases the stage volatility. For a consistent terminology, when correlation increases and overall volatility decreases, there is a negative chi (and negative vega) if ROV increases, and vice versa. There are numerous examples of both positive and negative vegas, and positive and negative chi's using this model.

---- *Figure 6 about here* ----

By setting $\rho_{VK_1} = \rho_{VK_2} = \rho_{VK_3} = \rho_{VK_4}$, Figure 6 illustrates the effects of a correlation change on the solution, using the base case value and cost volatilities. This reveals that the correlation decrease is accompanied by a rise in the volatility at all of the stages, $J = 1,2,3,4$, which leads to a fall in the parameter value and a rise in the effective mark-up factor. For the stages $J = 3,4$, although the volatility is observed to rise, there is a decrease in the ROV, indicating a positive chi (and positive vega). For stages 1 and 2, a rise in the volatility results in no change in the ROV (reflecting just the intrinsic value).

The effect of correlation changes on ROV is similar to the mixed and odd V and K vegas. Although correlation increases results in overall volatility decreases, sometimes the ROV increases and sometimes decreases, depending on the stage and on the level of correlation. For all stages, a decrease in correlation results in increased V thresholds.

Many of these results are changed for way out-of-the-money real sequential options, where $V < \hat{V}$, so the real option value is always equal to or above the intrinsic value (typically zero).

There are many other alternative combinations of changes in value volatility, investment cost volatility at each stage, and probability of failure at each stage that could be simulated, to illustrate the power and surprises of viewing sequential investment opportunities (and eventually investment requirements over stages) using this model.

4 Conclusion

We provide an analytical solution for a multi-factor, multi-phase sequential investment process, where there is the real option at any stage of continuing, or abandoning the project development. This model is particularly appropriate for real sequential R&D investment opportunities, such as geological exploration in natural resources that may be followed by development and then production, or drug development processes, where after drug discovery there are subsequent tests and trials required before production and marketing is feasible or allowed. Also, in these cases often there is a decreasing probability of project failure, as more information appears, and the efficacy and robustness of the original discovery are examined.

Other authors have provided unsatisfactory solutions to similar problems, or relied on bivariate or multivariate distribution functions, or required complex numerical solutions.

An advantage of our approach is that the effect of changing input parameter values can clearly be seen in terms of resulting overall project process volatility, the V thresholds which justify continuing with the investment stages, and on the ROV at each stage. Some of the results are intuitive. For instance, an increase in the failure probability by a constant amount for all stages , and increase in either value or investment cost volatility, or decrease in the correlation, results in raising the V threshold consistently although not necessarily proportionally. But these saem changes have sometimes intuitive and sometimes mixed and odd results on the ROV at each stage, and at different failure probability, volatility and correlation levels.

So in general, the effect of changes in input parameter values on the real option value and on the investment process continuance is sometimes surprising, and dependent on the specific input values and the number and sequence of stages, seen only in the solutions for each case. Indeed, the degree of real option moneyness matters in the magnitude (and even sign) of the sensitivity of real sequential investment options to changes in some critical parameter values.

Our model is not appropriate where the probability of failure increases with completion of each investment stage. Also we have assumed instantaneous investment completion, constant project value and investment cost drifts, volatilities and correlation, and no competition. Relaxing these assumptions are challenging issues for further research.

Appendix A: One-Stage Model

The function Q_1 (1.4) specifies an ellipse defined over a two-dimensional space spanned by two unknown parameters, β_1 and η_{10}. Since for a zero value of one parameter, the other parameter takes on a positive and a negative value, Q_1 has a presence in all 4 quadrants, which we label I – IV. The specification for these four quadrants is:

$$\text{I:} \quad \{\beta_{11}, \eta_{101}\} \quad \beta_{11} \geq 0, \eta_{101} \geq 0$$

$$\text{II:} \quad \{\beta_{12}, \eta_{102}\} \quad \beta_{12} \geq 0, \eta_{102} \leq 0$$

$$\text{III:} \quad \{\beta_{13}, \eta_{103}\} \quad \beta_{13} \leq 0, \eta_{103} \leq 0$$

$$\text{IV:} \quad \{\beta_{14}, \eta_{104}\} \quad \beta_{14} \leq 0, \eta_{104} \geq 0$$

This suggests that (1.3) takes the expanded form:

$$F_1 = A_{11} V^{\beta_{11}} K_1^{\eta_{111}} + A_{12} V^{\beta_{12}} K_1^{\eta_{112}} + A_{13} V^{\beta_{13}} K_1^{\eta_{113}} + A_{14} V^{\beta_{14}} K_1^{\eta_{114}} \tag{1.40}$$

Now, (1.40) is simplified by invoking the limiting boundary conditions. A justified economic incentive to exercise the option F_1 exists provided that the project value is sufficiently high and the investment cost is sufficiently low, and this incentive intensifies for project value increases and investment cost decreases. This suggests that the relevant quadrant is II, $\beta_{12} \geq 0$, $\eta_{112} \leq 0$ with $A_{12} > 0$. In contrast, no justified economic incentive exists if the project value is significantly low or the investment cost is significantly high. This suggests that quadrants I, III and IV should be ignored, that $A_{11} = A_{13} = A_{14} = 0$, and the corresponding option value is zero. This implies that (1.40) becomes:

$$F_1 = A_{12} V^{\beta_{12}} K_1^{\eta_{102}}. \tag{1.41}$$

Appendix B: Two-stage Model

The function Q_2 (1.14) specifies a hyper-ellipse defined over a three dimensional space spanned by the three unknown parameters, β_2, η_{21} and η_{22}. Since any one parameter has both a positive and a negative root for zero values of the remaining two parameters, the hyper-ellipse has a presence in all 8 quadrants. Labeling these quadrants as I – VIII, where:

I	$\{\beta_{21}, \eta_{211}, \eta_{221}\}$	$\beta_{21} \geq 0, \eta_{211} \geq 0, \eta_{221} \geq 0$
II	$\{\beta_{22}, \eta_{212}, \eta_{222}\}$	$\beta_{22} \geq 0, \eta_{212} \geq 0, \eta_{222} < 0$
III	$\{\beta_{23}, \eta_{213}, \eta_{223}\}$	$\beta_{23} \geq 0, \eta_{213} < 0, \eta_{223} \geq 0$
IV	$\{\beta_{24}, \eta_{214}, \eta_{224}\}$	$\beta_{24} \geq 0, \eta_{214} < 0, \eta_{224} < 0$
V	$\{\beta_{25}, \eta_{215}, \eta_{225}\}$	$\beta_{25} < 0, \eta_{215} \geq 0, \eta_{225} \geq 0$
VI	$\{\beta_{26}, \eta_{216}, \eta_{226}\}$	$\beta_{26} < 0, \eta_{216} \geq 0, \eta_{226} < 0$
VII	$\{\beta_{27}, \eta_{217}, \eta_{227}\}$	$\beta_{27} < 0, \eta_{217} < 0, \eta_{227} \geq 0$
VIII	$\{\beta_{28}, \eta_{218}, \eta_{228}\}$	$\beta_{28} < 0, \eta_{218} < 0, \eta_{228} < 0$

The expanded version of the valuation function (1.13) then becomes:

$$F_2 = \sum_{M=1}^{8} A_{2M} V^{\beta_{2M}} K_1^{\eta_{21M}} K_2^{\eta_{22M}}. \tag{1.42}$$

The form of (1.42) is simplified by invoking the limiting boundary conditions. Applying a similar argument as before reveals the relevant quadrant to be IV. Exercising the option F_2 is economically justified only if the project value V is sufficiently high and the investment expenditures, K_1 and K_2, are sufficiently low, while the resulting option value F_2 only becomes significantly high provided that $\beta_2 \geq 0$, $\eta_{21} < 0$ and $\eta_{22} < 0$. In contrast, there is no economic justification for exercising the option F_2 whenever the project value is sufficiently low, or either of the two investment expenditures, K_1 and K_2, are sufficiently high. This suggests that the quadrants other than IV are not relevant, and that their coefficients, A_{21}, A_{22}, A_{23}, A_{25}, A_{26}, A_{27} and A_{28}, are all set to equal zero. Consequently, (1.42) simplifies to:

$$F_2 = A_{24} V^{\beta_{24}} K_1^{\eta_{214}} K_2^{\eta_{224}}. \tag{1.43}$$

Appendix C: N-Stage Model

The value of the option to invest at the $J = N$ stage, denoted by $F_N = F_N\left(V, K_1, \ldots, K_N\right)$, is described by a $N+1$ dimensional partial differential equation, whose solution takes the form of a recursive product power function:

$$F_N = A_N V^{\beta_N} K_1^{\eta_{1N}} K_2^{\eta_{2N}} \ldots K_N^{\eta_{NN}} = B_N F_{N-1}^{\phi_N} K_N^{1-\phi_N}, \tag{1.44}$$

where the power parameters for $F_N\left(\cdot\right)$ are related to the ϕ_J according to:

$$\begin{aligned}
\beta_N &= \phi_N \phi_{N-1} \ldots \phi_3 \phi_2 \phi_1, \\
\eta_{1N} &= \phi_N \phi_{N-1} \ldots \phi_3 \phi_2 \left(1-\phi_1\right), \\
\eta_{2N} &= \phi_N \phi_{N-1} \ldots \phi_3 \left(1-\phi_2\right), \\
&\vdots \\
\eta_{N-1N} &= \phi_N \left(1-\phi_{N-1}\right), \\
\eta_{NN} &= \left(1-\phi_N\right).
\end{aligned}$$

Also, we have $B_N = A_N A_{N-1}^{-\phi_N}$. The stage-$N$ value matching relationship can now defined as:

$$B_N \hat{F}_{N-1,N}^{\phi_N} \hat{K}_{NN}^{1-\phi_N} = \hat{F}_{N-1,N} - \hat{K}_{NN} \tag{1.45}$$

where $\hat{V}_N, \hat{K}_{1N}, \ldots, \hat{K}_{NN}$ denote the respective optimal threshold levels with

$$\begin{aligned}
\hat{F}_{N-1,N} &= F_{N-1}\left(\hat{V}_N, \hat{K}_{1N}, \ldots, \hat{K}_{N-1N}\right) \\
&= A_{N-1} \hat{V}_N^{\beta_{N-1}} \hat{K}_{1N}^{\eta_{1N-1}} \hat{K}_{2N}^{\eta_{2N-1}} \ldots \hat{K}_{N-1N}^{\eta_{N-1N-1}}.
\end{aligned}$$

The stage-N value matching relationship (1.45) is expressed in the form of a two factor investment opportunity model, the value gained after exercise $F_{N-1,N}$ and the investment cost K_{NN}, so the thresholds can be determined from standard theoretical results. It follows that:

$$\hat{F}_{N-1,N} = A_{N-1} \hat{V}_N^{\beta_{N-1}} \hat{K}_{1N}^{\eta_{1N-1}} \hat{K}_{2N}^{\eta_{2N-1}} \ldots \hat{K}_{N-1N}^{\eta_{N-1N-1}} = \frac{\phi_N}{\phi_N - 1} \hat{K}_{NN} \tag{1.46}$$

and

$$\hat{V}_N = \left(\frac{\phi_N}{A_{N-1}\left(\phi_N - 1\right)} \right)^{\delta_{0N}} \hat{K}_{1N}^{\delta_{1N}} \hat{K}_{2N}^{\delta_{2N}} \ldots \hat{K}_{N-1N}^{\delta_{N-1N}} \hat{K}_{NN}^{\delta_{NN}} \tag{1.47}$$

where

$$\begin{aligned}
\delta_{0N} &= 1/\phi_{N-1} \cdots \phi_3 \phi_2 \phi_1 , \\
\delta_{1N} &= \left(\phi_1 - 1\right)/\phi_1 , \\
\delta_{2N} &= \left(\phi_2 - 1\right)/\phi_2 \phi_1 , \\
\delta_{3N} &= \left(\phi_3 - 1\right)/\phi_3 \phi_2 \phi_1 , \\
&\vdots \\
\delta_{N-1N} &= \left(\phi_{N-1} - 1\right)/\phi_{N-1} \cdots \phi_3 \phi_2 \phi_1 , \\
\delta_{NN} &= 1/\phi_{N-1} \cdots \phi_3 \phi_2 \phi_1 .
\end{aligned}$$

The parameter ϕ_N is evaluated from the characteristic root equation, which can be expressed as:

$$Q_N\left(\phi_N\right) = \tfrac{1}{2}\sigma_N^2 \phi_N \left(\phi_N - 1\right) + \phi_N \left(r + \lambda_{N-1} - \theta_{K_N}\right) - \left(r + \lambda_N - \theta_{K_N}\right) = 0. \tag{1.48}$$

The variance term σ_N^2 is given by:

$$\sigma_N^2 = \mathrm{w}^{\mathrm{T}} \Omega \mathrm{w} \tag{1.49}$$

where Ω is the $N+1$ dimensional square variance-covariance matrix with its first diagonal element being σ_V^2, the second $\sigma_{K_1}^2$, and so on until $\sigma_{K_N}^2$. The off-diagonal elements denote the corresponding covariances. The column vector w is given by:

$$\mathrm{w} = \begin{bmatrix} \phi_{N-1} \cdots \phi_3 \phi_2 \phi_1 \\ \phi_{N-1} \cdots \phi_3 \phi_2 \left(1 - \phi_1\right) \\ \phi_{N-1} \cdots \phi_3 \left(1 - \phi_2\right) \\ \vdots \\ \phi_{N-1}\left(1 - \phi_{N-2}\right) \\ \left(1 - \phi_{N-1}\right) \\ -1 \end{bmatrix}$$

The full expression for $\sigma_N^2 = \mathrm{w}^{\mathrm{T}}\Omega\mathrm{w}$ is:

$$\begin{aligned}
\tfrac{1}{2}\sigma_N^2 =\ & \tfrac{1}{2}\sigma_V^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_2^2\phi_1^2 + \tfrac{1}{2}\sigma_{K_1}^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_2^2\phi_2^2\left(1-\phi_1\right)^2 + \tfrac{1}{2}\sigma_{K_2}^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_3^2\left(1-\phi_2\right)^2 \\
& + \ldots + \tfrac{1}{2}\sigma_{K_{N-2}}^2\phi_{N-1}^2\left(1-\phi_{N-2}\right)^2 + \tfrac{1}{2}\sigma_{K_{N-1}}^2\left(1-\phi_{N-1}\right)^2 + \tfrac{1}{2}\sigma_{K_N}^2 \\
& + \rho_{VK_1}\sigma_V\sigma_{K_1}\phi_{N-1}^2\ldots\phi_3^2\phi_2^2\phi_1\left(1-\phi_1\right) + \rho_{VK_2}\sigma_V\sigma_{K_2}\phi_{N-1}^2\ldots\phi_3^2\phi_2\left(1-\phi_2\right)\phi_1 \\
& + \ldots + \rho_{VK_{N-1}}\sigma_V\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_2\phi_1 \\
& \qquad - \rho_{VK_N}\sigma_V\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_2\phi_1 \\
& + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\phi_{N-1}^2\ldots\phi_3^2\phi_2\left(1-\phi_2\right)\left(1-\phi_1\right) \\
& \quad + \rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\phi_{N-1}^2\ldots\phi_4^2\phi_3\left(1-\phi_3\right)\phi_2\left(1-\phi_1\right) \\
& + \ldots + \rho_{K_1K_{N-2}}\sigma_{K_1}\sigma_{K_{N-2}}\phi_{N-1}^2\phi_{N-2}\left(1-\phi_{N-2}\right)\phi_{N-3}\ldots\phi_2\left(1-\phi_1\right) \\
& \quad + \rho_{K_1K_{N-1}}\sigma_{K_1}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_2\left(1-\phi_1\right) \\
& \qquad - \rho_{K_1K_N}\sigma_{K_1}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_2\left(1-\phi_1\right) \\
& + \rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\phi_{N-1}^2\ldots\phi_4^2\phi_3\left(1-\phi_3\right)\left(1-\phi_2\right) \\
& \quad + \rho_{K_2K_4}\sigma_{K_2}\sigma_{K_4}\phi_{N-1}^2\ldots\phi_5^2\phi_4\left(1-\phi_4\right)\phi_3\left(1-\phi_2\right) \\
& + \ldots + \rho_{K_2K_{N-2}}\sigma_{K_2}\sigma_{K_{N-2}}\phi_{N-1}^2\phi_{N-2}\left(1-\phi_{N-2}\right)\phi_{N-3}\ldots\phi_3\left(1-\phi_2\right) \\
& \quad + \rho_{K_2K_{N-1}}\sigma_{K_2}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_3\left(1-\phi_2\right) \\
& \qquad - \rho_{K_2K_N}\sigma_{K_2}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_3\left(1-\phi_2\right) \\
& + \ldots \\
& + \rho_{K_{N-3}K_{N-2}}\sigma_{K_{N-3}}\sigma_{K_{N-2}}\phi_{N-1}^2\left(1-\phi_{N-2}\right)\left(1-\phi_{N-3}\right) \\
& \quad + \rho_{K_{N-3}K_{N-1}}\sigma_{K_{N-3}}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\left(1-\phi_{N-3}\right) \\
& \qquad - \rho_{K_{N-3}K_N}\sigma_{K_{N-3}}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\left(1-\phi_{N-3}\right) \\
& + \rho_{K_{N-2}K_{N-1}}\sigma_{K_{N-2}}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\left(1-\phi_{N-2}\right) \\
& \qquad - \rho_{K_{N-2}K_N}\sigma_{K_{N-2}}\sigma_{K_N}\phi_{N-1}\left(1-\phi_{N-2}\right) \\
& - \rho_{K_{N-1}K_N}\sigma_{K_{N-1}}\sigma_{K_N}\left(1-\phi_{N-1}\right),
\end{aligned}$$

Because of the homogeneity degree-one property, we have $\mathrm{w}^{\mathrm{T}}\mathbf{i} = 0$ where $\mathbf{i}$ is the unit vector. Note that for $N = 1$, $\mathrm{w}^{\mathrm{T}} = [1, -1]$.

Having evaluated ϕ_N, we can solve for:

$$B_N = \frac{\left(\phi_N - 1\right)^{\phi_N - 1}}{\phi_N^{\phi_N}}, \tag{1.50}$$

and then A_N is determined from:

$$A_N = B_N A_{N-1}^{\phi_N}. \tag{1.51}$$

The $A_{N-1} \ldots A_1$ and $B_{N-1} \ldots B_1$ are obtainable by applying (1.50) and (1.51) recursively, starting at $J = 1$ and ending at $J = N - 1$.

The solution to the stage-*N* investment decision is obtained through a process of backwardation, starting from the stage-one decision. This backwardation process yields consecutively the values of $\phi_1, \phi_2, \ldots, \phi_{N-1}$, $B_1, B_2, \ldots, B_{N-1}$ and $A_1, A_2, \ldots, A_{N-1}$, which are required for evaluating ϕ_N, B_N and A_N. From these values, we can then determine the discriminatory boundary linking the project value threshold $\hat{V}_N$ with the investment cost thresholds $\hat{K}_{1N}, \hat{K}_{2N}, \ldots \hat{K}_{NN}$. Further, for a meaningful solution to the stage-*N* investment decision to be obtained, the $\phi_1, \phi_2, \ldots, \phi_{N-1}, \phi_N$ have to individually exceed 1, which demands because of (1.49) that $\lambda_N > \lambda_{N-1} > \ldots > \lambda_2 > \lambda_1$.

Figure 1

Sequential Investment Process

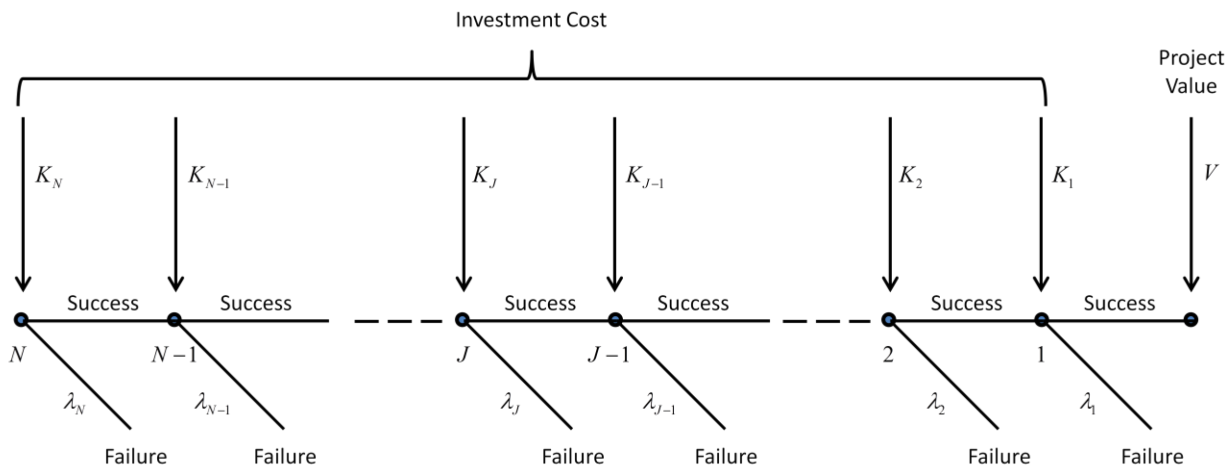


Figure 2

The Q_2 Function for Investment Stage 2

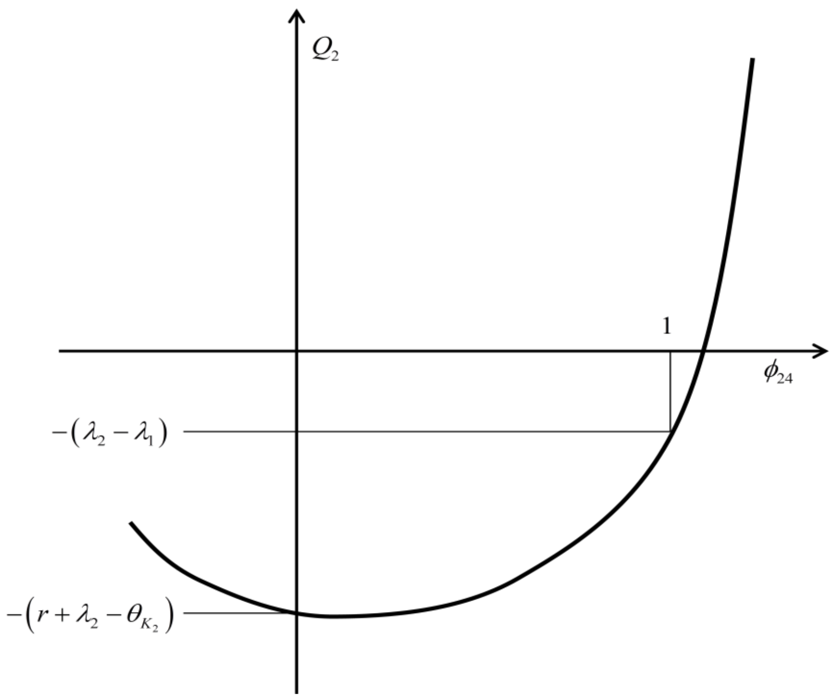


Table 1

Base Case Information

Project value drift rate	θ_V	0%
Project value volatility	σ_V	25%
Investment cost drift rate	$\theta_{K_1} = \theta_{K_2} = \theta_{K_3} = \theta_{K_4}$	0%
Investment cost volatility	$\sigma_{K_1} = \sigma_{K_2} = \sigma_{K_3} = \sigma_{K_4}$	5%
Stage 1 failure probability	λ_1	0%
Stage 2 failure probability	λ_2	10%
Stage 3 failure probability	λ_3	20%
Stage 4 failure probability	λ_4	40%
Risk-free probability	r	6%

All the correlations between the project value and the investment costs at each stage are set to equal zero, so the correlation matrix is specified by:

	V	K_1	K_2	K_3	K_4
V	100%				
K_1	0%	100%			
K_2	0%	0%	100%		
K_3	0%	0%	0%	100%	
K_4	0%	0%	0%	0%	100%

Table 2

Base Case Results

$V=100$, $K_1=K_2=K_3=K_4=25$

STAGE	VOLATILITY	ϕ	MEF	V^	V^-ΣK_N	ROV
1	0.2550	1.9478	2.0551	51.3766	26.3766	75.0000
2	0.4918	1.4294	1.8534	92.6715	42.6715	50.0000
3	0.7015	1.2176	1.6929	126.9680	51.9680	51.1388
4	0.8535	1.2760	1.2720	127.1951	27.1951	32.0015

Figure 3

The Effect of Increasing the Failure Probability by a Constant Amount of 5%

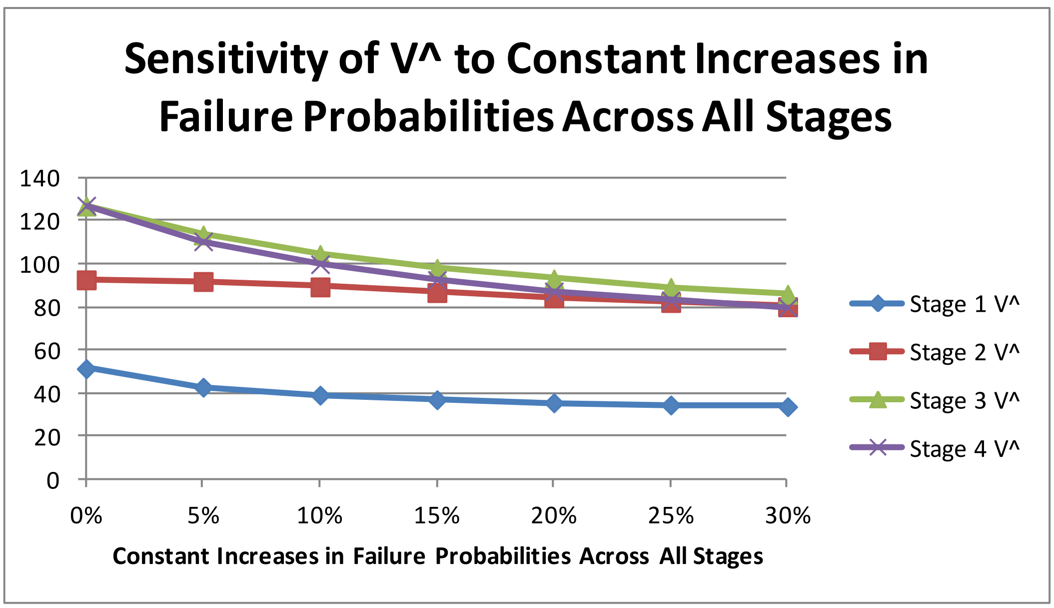


Figure 4

The Effect of Changing the Project Value Volatility

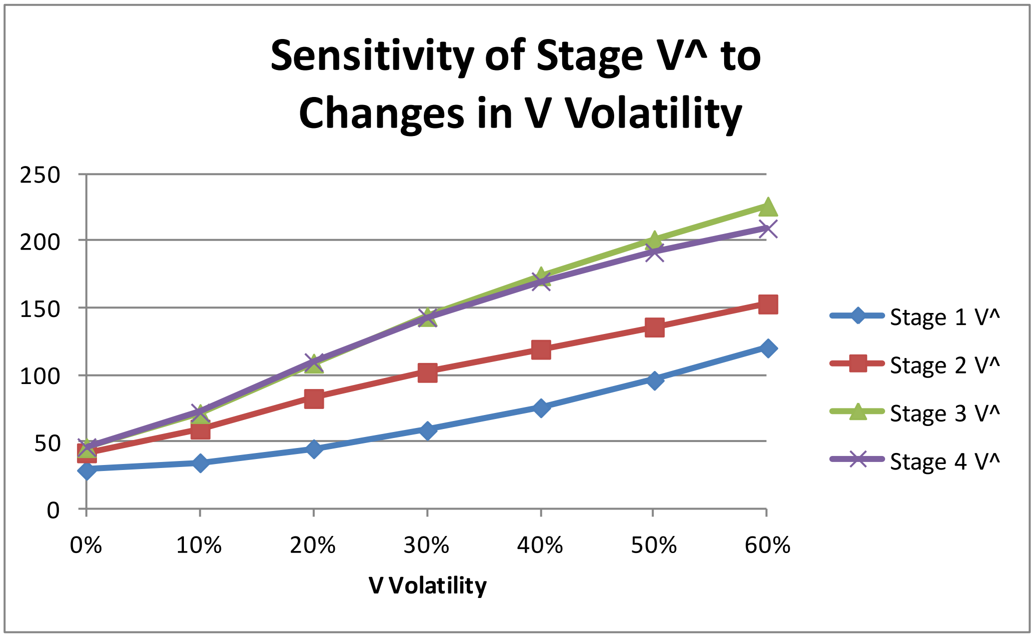


Figure 5

The Effect of Increasing all Investment Cost Volatilities to 10%

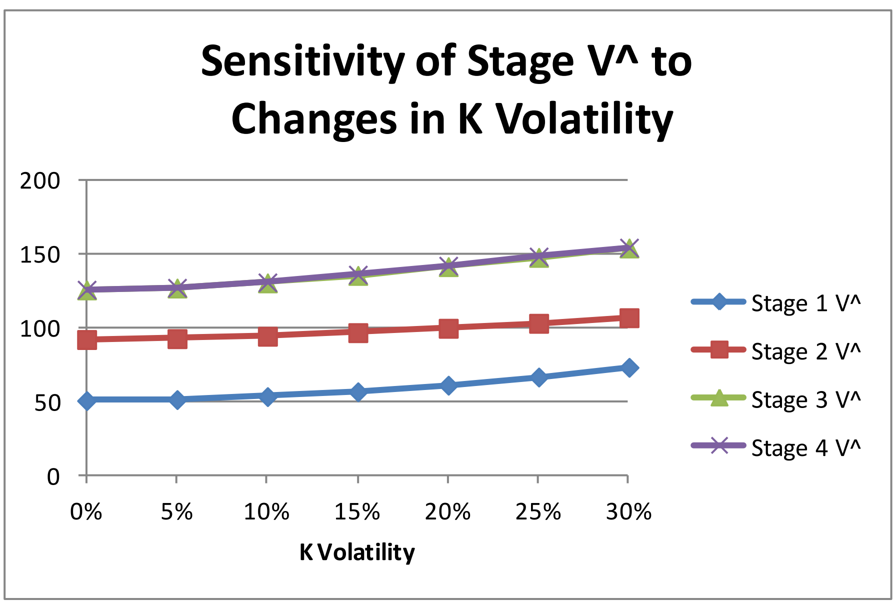
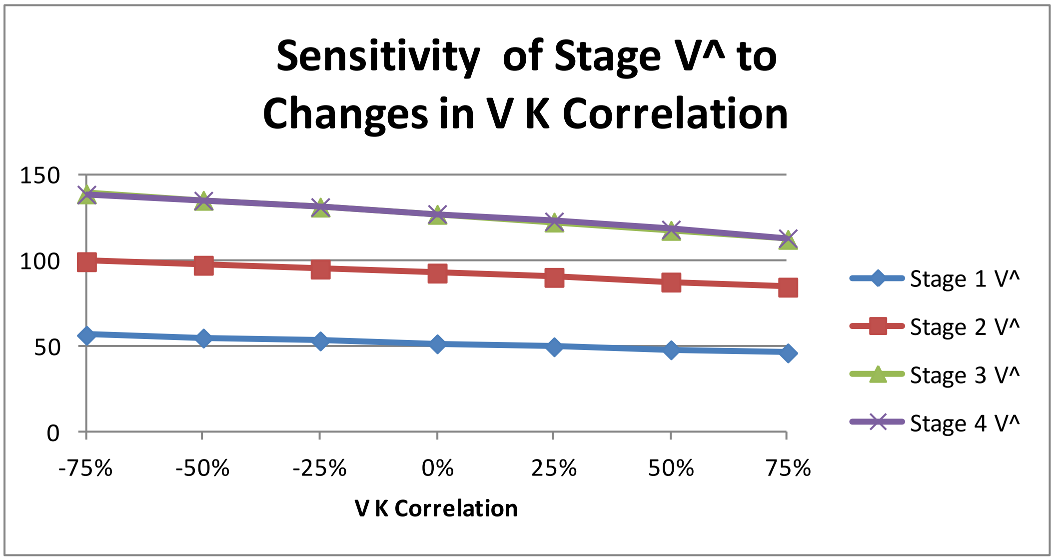


Figure 6

The Effect of Changing the Correlation between Value and Cost at All Stages



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