LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

MASTER OF SCIENCE IN ACTUARIAL SCIENCE

MASTERS FINAL WORK

DISSERTATION

ADJUSTMENT COEFFICIENT FOR EXCESS OF LOSS REINSURANCE WITH REINSTATEMENTS

MARIA ANGELICA OJEDA DAVILA

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Dedicated to Valentina

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ABSTRACT

In this dissertation we present a general procedure for the calculation of the adjustment coefficient of the retained risk for an excess of loss reinsurance with reinstatements, when there is no aggregate deductible, following the model studied by Sundt (1991).

We study how to calculate the initial reinsurance premium for this kind of contracts, when there is an aggregate layer, under different premium principles such as pure premium, expected value premium, standard deviation premium and the proportional hazard premium principles.

In order to calculate the insurer's adjustment coefficient we need to find the joint distribution of the insurer's total claims and the aggregate claims of the reinsurer, considering an excess of loss reinsurance contract without reinstatements. For this reason, we study the bivariate Panjer's recursion developed by Sundt (1999).

Numerical examples, including selection of the optimal layer, are discussed in this dissertation.

KEYWORDS: Excess of Loss Reinsurance with reinstatements, Initial Reinsurance premium, Adjustment coefficient, Bivariate Panjer's Recursion Formula, Optimal Layer.

RESUMO

Nesta dissertação propõe-se um método para o cálculo do coeficiente de ajustamento do risco retido para resseguro excedente de danos com reintegrações, quando não há dedutível agregado, seguindo o modelo analisado por Sundt (1991).

O prémio inicial de resseguro para este tipo de contratos é realizado sob diferentes princípios do calculo do prémio, tais como: principio do prémio puro, principio do valor esperado, principio do desvio padrão e principio da transformada PH “Proportional Hazard”.

Para o cálculo do coeficiente de ajustamento, precisamos de determinar a distribuição conjunta do total de indemnizações do segurador e total de indemnizações do ressegurador, para resseguro excedente de danos, sem reintegrações. Por esta razão, estudamos a fórmula recursiva bivariada do Panjer desenvolvido por Sundt (1999).

Apresentam-se exemplos numéricos, incluindo a seleção do ‘’layer” óptimo.

PALAVRAS-CHAVE: Resseguro Excedente de Danos, Prêmio Inicial de Resseguro, Coeficiente de Ajustamento, Formula Recursiva Bivariada de Sundt, “Layer” Óptimo.

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1. Introduction

The aim of this dissertation is to present a general method for calculating the adjustment coefficient of the retained risk for an excess of loss reinsurance with reinstatements, when there is no aggregate deductible. We consider that the reinstatements premiums are subjected to the same percentages regarding the premium initially paid.

Excess of loss reinsurance contracts are non-proportional treaties used commonly for catastrophe protection, which protect against single serious loss or losses. These contracts have a worldwide significance and there is hardly an insurer who does not use or write such reinsurance contracts.

Under an excess of loss contract with no aggregate limits the reinsurer covers the layer m xs l of each loss, i.e. when a loss $y$occurs, the reinsurer is liable for $\min(\max(0, y-l), m)$. But it is a common practice for a catastrophic contract that the layer has to be reinstated whenever a loss hits it.
Although there is no limit to the number of times that the layer is reinstated, it is usually reinstated only once, according to Klin R.and Klin S. (2001).

Regarding when the reinstatements premiums are due, Sundt (1991) considered that they are paid whenever the losses happen, in such a way that the full layer is reinstated, whereas Hess and Schmidt (2004) proposed an optimal reinsurance premium plan considering that the reinstatement premiums paid immediately after the layer is consumed. In both case, the reinstatements premiums are pro rata to the claim size to the layer. We will follow the model studied by Sundt (1991) throughout this thesis.

Sundt (1991) proposed the methodology to calculate the initial premium under the pure premium principle and the standard deviation principle for an excess of loss reinsurance with aggregate layer. Furthermore, Mata (2000) and Walhin and Paris (2001) priced the initial premium using the proportional hazard premium principle considering excess of loss reinsurance with reinstatements with no aggregate deductible.

In Chapter 2 we introduce the reinstatements in excess of loss reinsurance contracts, following the model studied by Sundt (1991). A numerical example is given to make it easier to understand this type of contracts.

In Chapter 3 we study the premiums and claims on excess of loss reinsurance contracts with reinstatements and explain the relationship between them. Also, we introduce the stop loss transform that is used throughout this thesis and played an important role during the calculations.

In Chapter 4 we study how to calculate the initial reinsurance premium for excess of loss reinsurance contracts with aggregate layer under different premium principles such as pure premium, expected value premium, standard deviation premium and the proportional hazard premium principles.

In the case of the standard deviation premium principle, in order to have real solutions for the initial premium a condition for the loading coefficient is given. This condition is studied in detail in Walhin and Paris (2001), Section3.

In Chapter 5 we analyse the equations that must be fulfilled to assure that the insurer's adjustment coefficient exists. Then, in Chapter 6 we show the procedure for calculating the insurer's adjustment coefficient, i.e. for the retained risk. We assume that there is no aggregate deductible.

In order to calculate the adjustment coefficient we need to find the joint distribution of the insurer total claims and the aggregate claims of the reinsurer, considering an excess of loss reinsurance contract without reinstatements. For this reason, in Chapter 7, we study the bivariate Panjer's recursion developed by Sundt (1999).

Numerical examples, including selection of the optimal layer, are discussed in Chapter 8 and some conclusions and comments on future research are set out in Chapter 9.All the calculations were made using the software `R`(2011)and the package `actuar` due to Dutang, Goulet and Pigeon (2008).

Much of the work of Chapters 2 and 3 of this thesis is presented in the paper by Sundt (1991).

2. Reinstatements in Excess of Loss Reinsurance Contracts

The pricing of this type of reinsurance was initially studied by Sundt (1991). He proposed the model and the methodology to calculate the initial premium under the pure premium and the standard deviation premium principles for the layer $m$in excess of $l(m$ xs $l)$ with aggregate deductible L with K reinstatements, that is the aggregate limit M is set equal to $(K+1)m$.

Considering the same notation as in Sundt (1991), we consider that X, the aggregate claim of the reinsurer portfolio for the layer m xs l, has a compound distribution, so that

$$X = \sum_{i=0}^{N} Z_i$$

where $Z_0 \equiv 0$, Z_i is the amount of the $i-th$ claim covered by the layer m xs l, that isZ_i is$\min\left(\max(0, Y_i - l), m\right)$and are independent and identically distributed with common distribution F.

$\{Y_i\}_{i=1,2,\dots}$ is a sequence of independent and identically distributed non-negative random variables, denoting the individual claims sizes, with common distributionF_0, independent of the number of claims N which is a counting random variable.

That is,

$$Z_i = \begin{cases} 0, & \text{if} \quad Y_i \leq l \\ Y_i, -l, & \text{if} \quad l < Y_i \leq m + l \\ m, & \text{if} \quad Y_i > m + l. \end{cases}$$

Then the cumulative distribution function of Z is

$$F(z) = \begin{cases} F_0(z+l), & \text{if } 0 \leq z < m \\ 1, & \text{if } \quad z \geq m. \end{cases}$$

Let G be the cumulative distribution function of X.

When the aggregate layer M xs L is considered, the aggregate ceded claims of the portfolio are $\min(\max(0, X-L), M)$ or expressed in the number of reinstatements $\min(\max(0, X-L), (K+1)m)$.

This is called an excess of loss reinsurance for the layer $m\ xs\ l\ xs\ L$ with K reinstatements.

The reinstatements may be free or paid. In case of free reinstatements, the premium is considered fixed because there is no extra premium to pay, but if the reinstatements are paid, every time that the claim hits the layer there is an extra premium to reinstate the layer so the ceding company has to pay a reinstatement premium, which is a percentage (c_k) of the premium initially paid by the insurer.

The $k-th$reinstatement premium,$c_k P$, is due when the aggregate claim is greater than $(k-1)m$, for $k = 1,2,\dots,K$, and paid pro rata to the claim size to the layer.

2.1. Numerical Example

Consider that we have an excess of loss reinsurance contract for the layer $200\ xs\ 70\ xs\ 100$ with 2 reinstatements. Moreover, the reinstatements premiums are paid at the following percentages $c_1 = 120\%$ and $c_2 = 150\%$, respectively.

Let us define the notation that will be used throughout this example.

Y_i is the amount of the $i-th$ claim.

$Z_i$is the amount of the $i-th$ claim covered by the layer 200 xs 70.

P is the initial reinsurance premium.

Table I shows the claim portfolio and which claims are considered inside each reinstatement, the aggregate limit and the aggregate deductible.

TABLE I

CLAIM PORTFOLIO

Claim number / Claim amounts	Agregate deductible		Aggregate limit							
			0 th reinstatement		1 st reinstatement			2 nd reinstatement		
	1	2	3	4	5	6	7	8	9	10
Y_i	120	120	130	210	100	140	170	160	180	120
Z_i	50	50	60	140	30	70	100	90	110	50
$Y_i - Z_i$	70	70	70	70	70	70	70	70	70	70

In this case, we have 10 claims and the aggregate claim of the portfolio is 1450. The aggregate claim of the reinsurer portfolio for the layer 200 xs 70 would be 750. However, considering that there is an aggregate layer of600 xs 100, the aggregate ceded claim is reduced to 600, i.e. 3 losses of 200. Therefore, the total claim retained is 850.

The first and the second claim will be paid by the insurer because the reinsurance contract has an aggregate deductible of 100. Then, starting from the third claim the reinsurer will reimburse to the insurer until a limit of 600, i.e. until the ninth claim. Finally, the last claim will be paid entirely by the insurer.

The reinsurer and the insurer agreed an initial premium P, that will cover only the $0-th$ reinstatements, but there will be extra premiums since the reinstatements are not free.

The first reinstatement, which is at 120%, will be paid after the third and the forth claim, so that the extra premium after the third claim would be $120\%P\,\frac{60}{200}$ and after the forth claim, $120\%P\,\frac{140}{200}$. Hence, the total extra premium for the first reinstatement is $120\%P$.

The second reinstatement, which is at 150%, will be paid after the fifth, sixth and seventh claim. Therefore, the extra premium after the fifth claim would be $150\%P\,\frac{30}{200}$, after the sixth claim $150\%P\,\frac{70}{200}$, and finally after the seventh claim $150\%P\,\frac{100}{200}$. Consequently, the total extra premium for the first reinstatement is $150\%P$.

After the eighth claim there is no reinstatement premium because the reinsurance contract has only 2 reinstatements and they were already paid, and the reinsurance contract finishes after the ninth claim. Finally, the total premium paid by the insurer is $370\%P$.

3. Premiums and Claims on Excess of Loss Reinsurance Contracts with Reinstatements

The cover under the $k-th$reinstatement $r_{L,k}$ is

$$r_{L,k} = \min\left(\max\left(0, X - km - L\right), m\right).$$

These random variables $r_{L,k}$ are greater than zero only if all the previous reinstatements are equal to the limit m.

Then the premium to be paid for the $k-th$ reinstatement, which is at c_k, pro rata to the claim size, is $c_k P \frac{r_{L,k-1}}{m}$.

The condition that the reinstatement is paid pro rata means that the premium for each reinstatement is a random variable. Then the total premium income, which is the initial premium plus the reinstatements premiums, is also a random variable, which is correlated with the aggregate claim.

Hence, the total premium income is

$$(1) \qquad T = P\left(1 + \frac{\sum_{k=1}^{K} c_k r_{L,k-1}}{m}\right)$$

and the aggregate claim payment for the aggregate deductible L with K reinstatements is

$$(2) \qquad R_{L,K} = \sum_{k=0}^{K} r_{L,k} = \min\left(\max(0, X - \mathrm{L}), (K+1)m\right).$$

There insurer net loss is given by subtracting the total premium income from the aggregate claim payment, that is $R_{L,K} - T$.

Let $\bar{G}$ be the stop loss transform defined as

$$\bar{G}(t) = \int_t^{\infty} (x-t)\,dG(x) = \int_t^{\infty} \big(1 - G(x)\big)dx$$

which is the expected value of $\max(0, X-t)$.

So, in terms of $\bar{G}$, the pure premium for the $k-th$ reinstatements

$$d_{L,k} = E\,(r_{L,k}) = \int_{L+km}^{L+(k+1)m} \big(1 - G(x)\big)dx = \bar{G}(L+km) - \bar{G}(L+(k+1)m)$$

and the pure premium for an excess of loss reinsurance for the layer $m$in excess of $l$with aggregate deductible L with K reinstatements is

$$D_{L,K} = E(R_{LK}) = \sum_{k=0}^{K} d_{L,k} = \int_{L}^{L+(K+1)m} \big(1 - G(x)\big)dx = \bar{G}(L) - \bar{G}(L+(K+1)m).$$

If the reinsurance contract has an aggregate unlimited cover, i.e. $K \to \infty$, the aggregate claim $R_{L,\infty}$ is $\max(0, X-L)$ and the pure premium $D_{L,\infty}$ would be $\bar{G}(L)$. Moreover, if it doesn't have an aggregate deductible the pure premium $D_{0,\infty}$ is just $E(X)$.

4. Calculation of the Initial Premium using Different Premium Principles

Sundt (1991) proposed the methodology to calculate the initial premium under the pure premium principle and the standard deviation principle.

Mata (2000) and Walhin and Paris (2001) priced the initial premium using the risk adjusted premium principle, considering as the distorted function the Proportional Hazard Transform, which satisfies some desirable properties studied in Wang (1996) and Silva and Centeno (1998).

Note that the papers by Mata (2000) and Walhin and Paris (2001) considered that there was no aggregate deductible.

4.1. Pure Premium Principle

According to the methodology presented by Sundt (1991), under this premium principle the initial premium should satisfy the equation

$$(3) \qquad E(T) = E(R_{L,K}).$$

This equation expresses the fact that the expected total reinsurance premium income should be equal to the expected ceded claims.

Inserting equations (1) and (2) in (3), the initial pure premium is

$$P = \frac{E\,(\sum_{k=0}^{K} r_{L,k})}{\left(1+\frac{E(\sum_{k=1}^{K} c_k r_{L,k-1})}{m}\right)}.$$

If the percentage c_k is the same for all the reinstatements, that is $c_k = c$; for $k = 1,2,\dots,K$. Then the initial premium P is:

$$P = \frac{E\,(R_{L,K})}{\left(1+c\frac{E(R_{L,K-1})}{m}\right)} = \frac{D_{L,K}}{\left(1+c\frac{D_{L,K-1}}{m}\right)}.$$

4.2. Expected Value Premium Principle

If instead of the pure premium calculation principle we use the expected value premium principle with loading coefficient α, i.e.

$$E(T) = (1+\alpha)E(R_{L,K}), \alpha \geq 0,$$

we get

$$P = (1+\alpha)\frac{E\,(\sum_{k=0}^{K} r_{L,k})}{\left(1+\frac{E(\sum_{k=1}^{K} c_k r_{L,k-1})}{m}\right)}$$

and when $c_k = c$

$$P = (1+\alpha)\frac{E\,(R_{L,K})}{\left(1+c\frac{E(R_{L,K-1})}{m}\right)} = (1+\alpha)\frac{D_{L,K}}{\left(1+c\frac{D_{L,K-1}}{m}\right)}\ .$$

4.3. Standard Deviation Premium Principle

For a risk U, the standard deviation premium principle is

$$P_U = E(U) + \gamma\sqrt{Var(U)},$$

where γ is the loading coefficient.

Since the total premium income itself is a random variable correlated with the claim payments on the reinstatement, Sundt (1991) apply this principle to calculate the initial premium using the equality

(4) $$E(T) = E\left(R_{L,K}\right) + \gamma\sqrt{Var(R_{L,K} - T)},\ \gamma \geq 0.$$

That is, the expected loading on the net risk is proportional to the standard deviation of the net risk.

Inserting (1) in (4), we get

$$E\left(P(1+\frac{\sum_{k=1}^{K} c_k r_{L,k-1}}{m})\right) = E\left(R_{L,K}\right) + \gamma\sqrt{Var(R_{L,K} - P(1+\frac{\sum_{k=1}^{K} c_k r_{L,k-1}}{m}))}.$$

This is equivalent to

$$pA = E\left(R_{L,K}\right) + \gamma\sqrt{Var\left(R_{L,K}\right) - 2pC + p^2B}$$

where

$p = P/m,$

$A = m + E(\sum_{k=1}^{K} c_k r_{L,k-1}),$

$B = Var(\sum_{k=1}^{K} c_k r_{L,k-1}),$

$C = Cov(\sum_{k=1}^{K} c_k r_{L,k-1}, R_{L,K})$.

It is important to note that in order to have real solutions of p, the discriminant has to be positive. Therefore, according to Walhin and Paris (2001), the loading coefficient γ should satisfy the following condition:

$$\gamma < \sqrt{\frac{A^2 Var(R_{L,K}) + B(E^2(R_{L,K})) - 2CAE(R_{L,K})}{BVar(R_{L,K}) - C^2}}.$$

Solving the quadratic equation, the initial premium $P$is

$$P = \frac{AE(R_{L,K}) - \gamma^2 C + \sqrt{(AE(R_{L,K}) - \gamma^2 C)^2 - (A^2 - \gamma^2 B)(E^2(R_{L,K}) - \gamma^2 Var(R_{L,K}))}}{A^2 - \gamma^2 B} * m.$$

4.4. Proportional Hazard (PH) Premium Principle

In general the initial premium should satisfy the condition that

(5) $$E_g(T) = E_g(R_{L,K}).$$

This equation expresses the fact that the distorted expected total premium income should be equal to the distorted expected claim payments, where g is a distortion function defined as $g(x) = x^{1/\rho}$ and ρ is called the risk aversion index. For a risk X, the premium calculated according to the PH principle is

$$E_g(X) = \int_0^\infty (1 - G(x))^{1/\rho} dx,\ \rho \geq 1.$$

Inserting equations (1) and (2) in (5), the initial risk adjusted premium is

$$P = \frac{E_g\ (\sum_{k=0}^{K} r_{L,k})}{\left(1 + \frac{E_g(\sum_{k=1}^{K} c_k r_{L,k-1})}{m}\right)}.$$

We should note that when the risk aversion index is equal to 1, this initial premium is the same that the one calculated under the pure premium principle.

If the percentage c_k is the same for all the reinstatements, that is $c_k = c,$ then the initial risk adjusted premium P is:

$$P = \frac{E_g\ (R_{L,K})}{\left(1+c\frac{E_g(R_{L,K-1})}{m}\right)} = \frac{D\text{mod}_{L,K}}{\left(1+c\frac{D\text{mod}_{L,K-1}}{m}\right)}.$$

Here, the premium for an excess of loss reinsurance for the layer $m$in excess of $l$with aggregate deductible L with K reinstatements is

$$D\text{mod}_{L,K} = E_g\left(R_{L,K}\right) = \int_{L}^{L+(K+1)m} \left(1 - G(x)\right)^{1/\rho} dx$$

$$= \bar{G}\text{mod}(L) - \bar{G}\text{mod}(L + (K+1)m)$$

where

$$\bar{G}\text{mod}(t) = \int_t^{\infty} \left(1 - G(x)\right)^{1/\rho} dx.$$

5. Adjustment Coefficient

From the point of view of the insurer, the adjustment coefficient R is defined as the unique positive solution of

$$E\left[e^{R(\tilde{S}-\tilde{c})}\right] = 1$$

where $\tilde{S}$ is the retained claims and $\tilde{c}$is the net premium of the insurer, if such a root exists.

We will assume that the moment generating function of $\tilde{S} - \tilde{c}$ exists for values of $r < \eta$ and the limit of the moment generating function of $\tilde{S} - \tilde{c}$ goes to infinity when $r \to \eta$.

Furthermore, it can be proved that the adjustment coefficient exists if and only of the insurer expected net profit is positive, i.e.

$$E(\tilde{c} - \tilde{S}) > 0. \tag{6}$$

The importance of the adjustment coefficient is because this value is used to calculate the ruin probability; the Lundberg's inequality shows this relation. See Klugman et al (2008), page 283.

Hence, the larger adjustment coefficient the smaller upper bound of the ultimate ruin probability.

6. Calculation of the Adjustment Coefficient for the Insurer

We will assume in this section that there is no aggregate deductible, i.e. $L = 0$. We also assume that all the reinstatements premiums are subject to the same percentages regarding the premium initially paid, i.e. $c_k = c$.

Let

$A_i = Y_i - Z_i$,

where A_i is the retained part of the $i-th$ claim, Y_i is the individual claim size and Z_i is the amount of the $i-th$ claim covered by the layer m xs l, that is Z_i is

$\min(\max(0, Y_i - l), m)$.Hence, letting$S = \sum_{i=0}^{N} Y_i$, then the insurer total claims if there was no aggregate limit, would be

$\sum_{i=0}^{N} A_i = \sum_{i=0}^{N} Y_i - \sum_{i=0}^{N} Z_i = S - X = W$.

Consequently, if we have an aggregate limit $M = (K+1)m$, the total retained claims $\tilde{S}$ will be

$$\tilde{S} = \sum_{i=0}^{N} A_i + \max(0, X - (K+1)m)$$

$$= S - X + \max(0, X - (K+1)m).$$

Then, the expected value of the retained claims is

$$E(\tilde{S}) = E(S) - E(X) + \bar{G}((K+1)m).$$

Assuming that the total premium income of the insurer is $\tilde{P}$, the net premium is

$$\tilde{c} = \tilde{P} - T = \tilde{P} - P\left(1 + c\frac{R_{0,K-1}}{m}\right)$$

and the expected net profit $E(\tilde{c} - \tilde{S})$, should be positive for the adjustment coefficient of the retained risk to exist. So,

$E\left[\tilde{P} - P\left(1 + c\frac{R_{0,K-1}}{m}\right) - (S - X + \max(0, X - (K+1)m))\right] > 0$ which is equivalent to $\tilde{P} - P(1 + c\frac{E(R_{0,K-1})}{m}) - (E(S) - E(X) + \bar{G}((K+1)m)) > 0.$

Then, the adjustment coefficient of the retained risk is the only positive root of

$E\left[e^{R(S - X + \max(0, X-(K+1)m) - (\tilde{P} - P(1 + c\frac{\min(X, Km)}{m})))}\right] = 1$, assuming that (6) is fulfilled.

To calculate $E\left[e^{R\,(S-X+\max(0,X-(K+1)m)-(\tilde{P}-P(1+c\frac{\min(X,Km)}{m})))}\right]$, we need to find the distribution of $S-X+\max(0,X-(K+1)m)+cP\frac{\min(X,Km)}{m}+P-\tilde{P}$, which is a function of $S-X$ and X. Moreover these variables are not independent.

Consequently, first it is necessary to determine the joint distribution of $(S-X,X)=(W,X)=(\sum_{i=0}^{N}A_i\,,\sum_{i=0}^{N}Z_i)$. We will use for thatthe bivariate Panjer's formula developed by Sundt (1999).

7. Bivariate Panjer's Recursion Formula

We recall that the univariate Panjer's recursion formula is

$$g_X(x)=Pr(X=x)=\frac{1}{1-af_Z(0)}\sum_{j=1}^{x}\left(a+b\frac{j}{x}\right)f_Z(j)g_X(x-j)\,,\qquad x=1,2,3,\ldots$$

$$g_X(0)=Pr(X=0)=P_N(f_Z(0))$$

when N belongs to the $(a,b,0)$ class and Z takes values on the non negative integers.

That is, the frequency distribution of the number of claims satisfies

$$Pr(N=k)=(a+\frac{b}{n})Pr(N=k-1),\qquad k=1,2,3,\ldots.$$

And the probability generating function is $P_N(z)=E\,(z^N)$.

In the particular case where N is Poisson distributed with parameter λ, ($a=0$ and $b=\lambda$), the recursion formula is

$$g_x=Pr(X=x)=\lambda\sum_{j=1}^{x}\left(\frac{j}{x}\right)f_Z(j)g_X(x-j)\,,\qquad x=1,2,3,\ldots$$

$$g_0 = g(X=0) = P_N\big(f_z(0)\big) = e^{-\lambda(1-f_Z(0))}$$

Let

$$(W,X) = (\sum_{i=1}^{N} A_i\,, \sum_{i=1}^{N} Z_i)$$

and assume that N belongs to the $(a,b,0)$ class, where (A_i, Z_i) is independent of the number of claims N. A_i and Z_i are not necessarily independent.

The following notation will be used

$$\sum_{x1,x2}^{s1,s2} w(x1,x2) = \sum_{x1=0}^{s1}\sum_{x2=0}^{s2} w(x1,x2) - w(0,0)$$

By assumption, $w(0,0) = 0$. So, we are going to consider

$$\sum_{x1,x2}^{s1,s2} w(x1,x2) = \sum_{x1=0}^{s1}\sum_{x2=0}^{s2} w(x1,x2).$$

Then, the recursion formula is

$$f_{W,X}(w,x) = \frac{1}{1-af_{A,Z}(0,0)}\sum_{u=0}^{w}\sum_{v=0}^{x}\left(a+b\frac{u}{w}\right)f_{W,X}(w-u,x-v)f_{A,Z}(u,v) \qquad w = 1,2,\ldots$$

$$x = 0,1,2,\ldots$$

$$f_{W,X}(w,x) = \frac{1}{1-af_{A,Z}(0,0)}\sum_{u=0}^{w}\sum_{v=0}^{x}\left(a+b\frac{v}{x}\right)f_{W,X}(w-u,x-v)f_{A,Z}(u,v) \qquad w = 0,1,2,\ldots$$

$$x = 1,2,\ldots$$

$$f_{W,X}(0,0) = f_{W,X}(W=0, X=0) = P_N(f_{A,Z}(0,0)).$$

Since $X > 0$ only if $W > 0$, we only need the first recursion, and

$f_{A,Z}(0,0) = f_A(0)$.

In the particular case when N is Poisson distributed with parameter λ, the recursion formula is

$$f_{W,X}(w,x) = \frac{\lambda}{w}\sum_{u=0}^{w}\sum_{v=0}^{x} u f_{W,X}(w-u, x-v) f_{A,Z}(u,v) \qquad w = 1,2,\ldots$$

$$x = 0,1,2,\ldots$$

$$f_{W,X}(0,0) = P_N(f_{A,Z}(0,0)) = e^{-\lambda(1-f_A(0))}.$$

8. Examples

We consider, as the example used by Witdouck, S. and Walhin, J. (2003),that the number of claims N is Poisson distributed with parameter 1.5 andY_i's are limited Pareto distributed defined as follows:

$$F_0(y) = \begin{cases} 0, & \text{if} \quad y \le 5 \\ \dfrac{5^{-1.5} - y^{-1.5}}{5^{-1.5} - 150^{-1.5}}, & \text{if} \quad 5 < y \le 150 \\ 1, & \text{if} \quad y > 150 \end{cases}$$

We consider an excess of loss reinsurance contract with layer $100\ xs\ 50$ with one paid reinstatement and $c_1 = 100\%$.

Note that we don't assume that the individual claim sizes are distributed as Pareto because it has no moment generating function, then the adjustment coefficient does not exists. Consequently, a limited Pareto is used instead.

Let us consider two cases:

In the first we consider $\tilde{P} = 23.13086$, which was calculated according to the expected value principle with loading coefficient 0.25, and the initial reinsurance premium is calculated according to the expected value principle with $\hat{\alpha} = 0.5$.

In the second we consider$\tilde{P} = 23.07642$, which was calculated according to the risk adjusted premium principle with risk aversion index 1.2, and the initial reinsurance premium is calculated according to the risk adjusted premium principle with $\hat{\rho} = 1.5$.

To be able of applying the bivariate formula, we started by discretizing the limited Pareto using the local moment matching method that matches the first moment, considering a step of discretisation$h = 5$. See Klugman et al (2008), pages 232-234.

The expected value of the aggregate claims is $E(S) = E(N)E(Y) = 18.5046$, and the expected value of the aggregate claims of the reinsurer for the layer 100 xs 50 is $E(X) = E(N)E(Z) = 1.098619$.

The total claims reinsured $\hat{S}$ is

$$\hat{S} = \min(X, 200) = R_{0,1},$$

the total claims retained $\tilde{S}$ is

$$\tilde{S} = S - X + \max(0, X - 200),$$

and its expected value is

$$E(\tilde{S}) = E(S) - E(X) + \bar{G}(200) = 17.40607.$$

For the first case

The expected value of claims reinsured is

$E(\hat{S}) = E(R_{0,1}) = \bar{G}(0) - \bar{G}(200) = 1.098617,$

and the initial reinsurance premium is

$$P = (1+0.5)\frac{E\,(R_{0,1})}{\left(1+\frac{E(R_{0,0})}{100}\right)} = (1+0.5)\frac{D_{0,1}}{\left(1+\frac{D_{0,0}}{100}\right)} = 1.630053.$$

The total premium income of the reinsurer is

$T = P\left(1+\frac{R_{0,0}}{100}\right) = 1.630053\left(1+\frac{R_{0,0}}{100}\right),$

then the expected total premium income of the reinsurer is

$E(T) = P\left(1+\frac{E\,(R_{0,0})}{100}\right) = (1+0.5)\frac{E\,(R_{0,1})}{\left(1+\frac{E(R_{0,0})}{100}\right)}\left(1+\frac{E\,(R_{0,0})}{100}\right) = (1+0.5)E\,(R_{0,1}) = 1.647925.$

Consequently, the net premium is:

$\tilde{c} = \tilde{P} - T = 23.13086 - 1.630053\left(1+\frac{R_{0,0}}{100}\right).$

To verify that the adjustment coefficient of the retained risk exists, first we calculate the net profit, $E(\tilde{c} - \tilde{S})$, and verify it is positive, i.e.

$$E\left[23.13086 - 1.630053\left(1+\frac{R_{0,0}}{100}\right) - (S - X + \max(0, X-200))\right] > 0$$

$$23.13086 - 1.5E\,(R_{0,1}) - (E(S) - E(X) + \bar{G}(200)) > 0$$

$$23.13086 - 1.647925 - 17.40607 = 4.076864 > 0.$$

Then, the adjustment coefficient of the retained risk is the only positive root of

(7) $$E\left[e^{R(S-X+\max(0,X-200)+0.01630053*\min(X,100)-21.50081)}\right] = 1.$$

For the second case

The expected value of claims reinsured is

$E_{\hat{g}}(\hat{S}) = E_{\hat{g}}(R_{0,1}) = \bar{G}\mathrm{mod1}(0) - \bar{G}\mathrm{mod1}(200) = 4.551078,$

where $\hat{g}$ is a distortion function defined as $\hat{g}(x) = x^{1/1.5}$.

The initial reinsurance premium is

$$P = \frac{E\hat{g}\ (R_{0,1})}{\left(1+\frac{E\hat{g}(R_{0,0})}{100}\right)} = \frac{D\text{mod}_{0,1}}{\left(1+\frac{D\text{mod}_{0,0}}{100}\right)} = \frac{\bar{G}\text{mod}(0)-\bar{G}\text{mod}(200)}{\left(1+\frac{\bar{G}\text{mod}(0)-\bar{G}\text{mod}(100)}{100}\right)} = 4.355717,$$

where $\bar{G}\text{mod}(t)$ is defined as

$$\bar{G}\text{mod}(t) = \int_t^{\infty}\left(1-G(x)\right)^{1/1.5}dx.$$

The total premium income of the reinsurer is

$T = P\left(1+\frac{R_{0,0}}{100}\right) = 4.355717\left(1+\frac{R_{0,0}}{100}\right),$

then the expected total premium income of the reinsurer is

$E(T) = P\left(1+\frac{E\left(R_{0,0}\right)}{100}\right) = 4.355717\left(1+\frac{E\left(R_{0,0}\right)}{100}\right) = 4.403475.$

Consequently, the net premium is:

$\tilde{c} = \tilde{P} - T = 23.0762 - 4.355717\left(1+\frac{R_{0,0}}{100}\right).$

To verify that the adjustment coefficient of the retained risk exists, first we calculate the net profit, $E\left(\tilde{c}-\tilde{S}\right)$, and verify it is positive, i.e.

$$E\left[23.0762 - 4.355717\left(1+\frac{R_{0,0}}{100}\right) - (S - X + \max(0, X-200))\right] > 0$$

$$23.0762 - 4.355717\left(1+\frac{E\left(R_{0,0}\right)}{100}\right) - (E(S) - E(X) + \bar{G}(200) > 0$$

$$23.07642 - 4.403475 - 17.40607 = 1.2668 > 0$$

Then, the adjustment coefficient of the retained risk is the only positive root of

(8) $$E\left[e^{R(S-X+\max(0,X-200)+0.04355717*\min(X,100)-18.72071)}\right] = 1.$$

In both cases, we need to calculate the joint distribution of $(S-X, X)$.

So, applying the bivariate Panjer's formula, We get that the recursion formula is:

$$f_{W,X}(0,0) = P_N(f_{A,Z}(0,0)) = e^{-\lambda(1-f_A(0))} = e^{-\lambda(1-0)} = 0.2231$$

$$f^*_{W,X}(w,x) = \frac{\lambda}{w}\left[\sum_{u=1}^{\min(w,9)} u f^*_{W,X}(w-u, x-0) f^*_{A,Z}(u,0) + \sum_{v=0}^{\min(x,20)} 10 f^*_{W,X}(w-10, x-v) f^*_{A,Z}(10,v)\right]; \qquad w = 1,2,\ldots \quad x = 0,1,2,\ldots$$

Where $f^*_{A,Z}$ is the probability function of (A,Z)on the new unit measure (the old measure/h) and $f^*_{W,X}$ is the probability function of (W,X)on the new unit measure(the old measure/h), where $h = 5$.

Tables II and III show the distributions of $f^*_{A,Z}$ and $f^*_{W,X}$, respectively:

TABLE II

$f^*_{A,Z}$

		Z^*					
		0	**1**	**2**	**3**	**…**	**20**
A^*	**0**	0	0	0	0	…	0
	1	P(Y=5)	0	0	0	…	0
	2	P(Y=10)	0	0	0	…	0
	3	P(Y=15)	0	0	0	…	0
	4	P(Y=20)	0	0	0	…	0
	5	P(Y=25)	0	0	0	…	0
	6	P(Y=30)	0	0	0	…	0
	7	P(Y=35)	0	0	0	…	0
	8	P(Y=40)	0	0	0	…	0
	9	P(Y=45)	0	0	0	…	0
	10	P(Y=50)	P(Y=55)	P(Y=60)	P(Y=65)	…	P(Y=150)

TABLE III

$f^*_{W,X}$

W^* \ X^*	0	1	2	3	…	20	21	22	23	24	…	40	41	42	43	44	45	…
0	2.23E-01	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
1	1.39E-01	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
2	1.53E-01	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
3	1.13E-01	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
4	8.85E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
5	6.55E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
6	4.86E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
7	3.60E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
8	2.69E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
9	2.03E-02	0	0	0	…	0	0	0	0	0	…	0	0	0	0	0	0	…
10	1.56E-02	1.27E-03	1.02E-03	8.33E-04	…	5.27E-05	0	0	0	0	…	0	0	0	0	0	0	…
11	1.08E-02	7.92E-04	6.36E-04	5.20E-04	…	3.29E-05	0	0	0	0	…	0	0	0	0	0	0	…
12	7.72E-03	8.71E-04	7.00E-04	5.73E-04	…	3.62E-05	0	0	0	0	…	0	0	0	0	0	0	…
13	5.27E-03	6.42E-04	5.16E-04	4.22E-04	…	2.67E-05	0	0	0	0	…	0	0	0	0	0	0	…
14	3.60E-03	5.03E-04	4.04E-04	3.30E-04	…	2.09E-05	0	0	0	0	…	0	0	0	0	0	0	…
15	2.44E-03	3.72E-04	2.99E-04	2.44E-04	…	1.55E-05	0	0	0	0	…	0	0	0	0	0	0	…
16	1.65E-03	2.76E-04	2.22E-04	1.81E-04	…	1.15E-05	0	0	0	0	…	0	0	0	0	0	0	…
17	1.11E-03	2.04E-04	1.64E-04	1.34E-04	…	8.50E-06	0	0	0	0	…	0	0	0	0	0	0	…
18	7.46E-04	1.53E-04	1.23E-04	1.00E-04	…	6.36E-06	0	0	0	0	…	0	0	0	0	0	0	…
19	4.99E-04	1.15E-04	9.28E-05	7.59E-05	…	4.80E-06	0	0	0	0	…	0	0	0	0	0	0	…
20	3.31E-04	8.84E-05	7.46E-05	6.39E-05	…	7.93E-06	3.73E-06	3.02E-06	2.46E-06	2.02E-06	…	6.22E-09	0	0	0	0	0	…
21	2.15E-04	6.15E-05	5.16E-05	4.40E-05	…	5.21E-06	2.33E-06	1.89E-06	1.54E-06	1.26E-06	…	3.89E-09	0	0	0	0	0	…
22	1.40E-04	4.38E-05	3.77E-05	3.28E-05	…	4.75E-06	2.56E-06	2.08E-06	1.69E-06	1.39E-06	…	4.28E-09	0	0	0	0	0	…
23	9.00E-05	2.99E-05	2.58E-05	2.26E-05	…	3.40E-06	1.89E-06	1.53E-06	1.25E-06	1.02E-06	…	3.15E-09	0	0	0	0	0	…
24	5.78E-05	2.04E-05	1.79E-05	1.57E-05	…	2.54E-06	1.48E-06	1.20E-06	9.78E-07	8.00E-07	…	2.47E-09	0	0	0	0	0	…
25	3.69E-05	1.38E-05	1.22E-05	1.08E-05	…	1.82E-06	1.09E-06	8.87E-07	7.23E-07	5.92E-07	…	1.83E-09	0	0	0	0	0	…
26	2.35E-05	9.35E-06	8.30E-06	7.41E-06	…	1.31E-06	8.11E-07	6.58E-07	5.36E-07	4.39E-07	…	1.35E-09	0	0	0	0	0	…
27	1.49E-05	6.30E-06	5.65E-06	5.08E-06	…	9.48E-07	6.01E-07	4.88E-07	3.98E-07	3.25E-07	…	1.00E-09	0	0	0	0	0	…
28	9.43E-06	4.24E-06	3.84E-06	3.48E-06	…	6.89E-07	4.49E-07	3.64E-07	2.97E-07	2.43E-07	…	7.51E-10	0	0	0	0	0	…
29	5.93E-06	2.83E-06	2.60E-06	2.39E-06	…	5.05E-07	3.40E-07	2.75E-07	2.25E-07	1.84E-07	…	5.67E-10	0	0	0	0	0	…
30	3.71E-06	1.88E-06	1.76E-06	1.64E-06	…	4.41E-07	3.25E-07	2.73E-07	2.31E-07	1.95E-07	…	6.20E-09	4.58E-09	3.62E-09	2.85E-09	2.24E-09	1.75E-09	…
31	2.30E-06	1.22E-06	1.16E-06	1.09E-06	…	2.99E-07	2.21E-07	1.86E-07	1.56E-07	1.32E-07	…	3.90E-09	2.87E-09	2.26E-09	1.78E-09	1.40E-09	1.10E-09	…
32	1.43E-06	7.94E-07	7.63E-07	7.27E-07	…	2.26E-07	1.74E-07	1.47E-07	1.26E-07	1.07E-07	…	4.18E-09	3.15E-09	2.49E-09	1.96E-09	1.54E-09	1.21E-09	…
33	8.80E-07	5.11E-07	4.95E-07	4.76E-07	…	1.55E-07	1.21E-07	1.03E-07	8.79E-08	7.51E-08	…	3.07E-09	2.32E-09	1.83E-09	1.44E-09	1.14E-09	8.88E-10	…
34	5.41E-07	3.28E-07	3.22E-07	3.11E-07	…	1.09E-07	8.59E-08	7.36E-08	6.31E-08	5.41E-08	…	2.39E-09	1.82E-09	1.44E-09	1.13E-09	8.89E-10	6.96E-10	…
⋮	⋮	⋮	⋮	⋮	…	⋮	⋮	⋮	⋮	⋮	…	⋮	⋮	⋮	⋮	⋮	⋮	…

The survival functions were calculated until they reached the value 10^{-10}. Finally, after solving equations (7) and (8) to get the adjustment coefficient of the retained risk, we found that these values for the first and the second case are 0.018839 and 0.006708, respectively.

These examples show how to apply the general procedure to calculate the adjustment coefficient of the insurer independent of the premium principle used to calculate the initial reinsurance premium.

Next section studies the selection of the insurer's optimal layer for each case.

8.1. Optimal Layer

In both cases, when the initial reinsurance premium is calculated under the expected value premium principle and the PH premium principle, we analyse the insurer's adjustment coefficient when the deductible changes.

We consider the following layers $100\,xs\,l$ such that deductible$l = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.$

Figures1 and 2 show how the net profit and the insurer's adjustment coefficient change when different deductibles are considered. In this case the initial reinsurance premium is calculated according to the expected value principle.

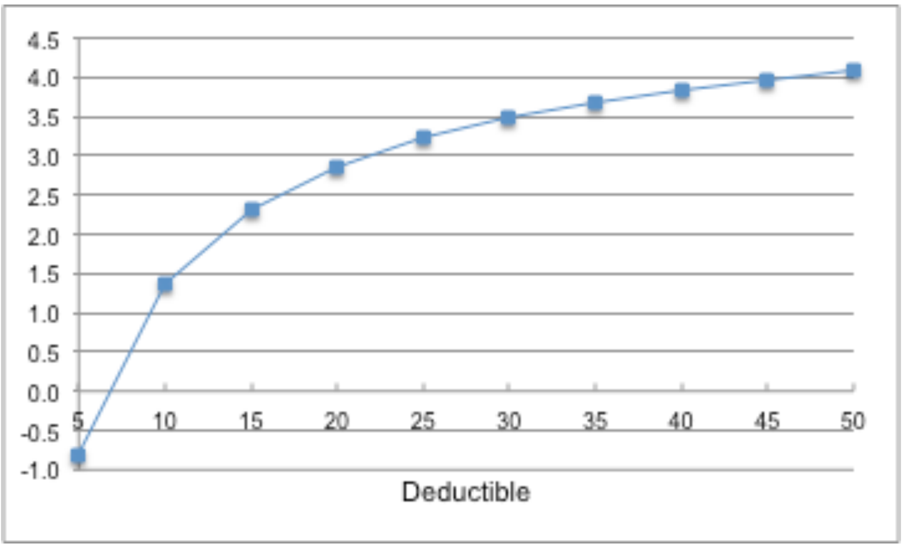

FIGURE 1 - INSURER'S NET PROFIT FOR CASE 1

We found that the insurer's adjustment coefficient exists only for deductibles greater or equal than 10, since the insurer's net profits are positive starting from that value, considering only the possible layers specified before.

The deductible that maximizes the insurer's adjustment coefficient is 15. Therefore, the optimal layer is $100\ xs\ 15$.

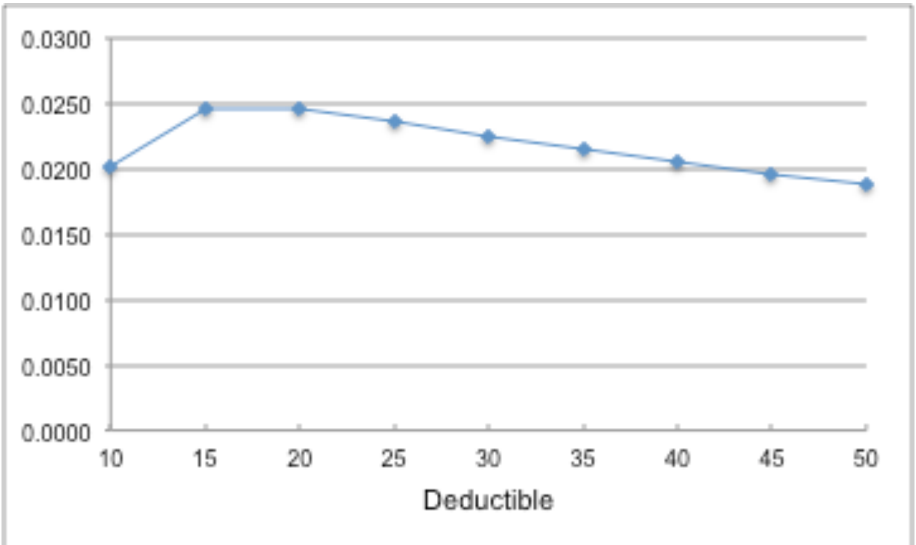

FIGURE 2- INSURER'S ADJUSTMENT COEFFICIENT FOR CASE 1

For the second case, when the initial reinsurance premium is calculated according to the PH premium principle.

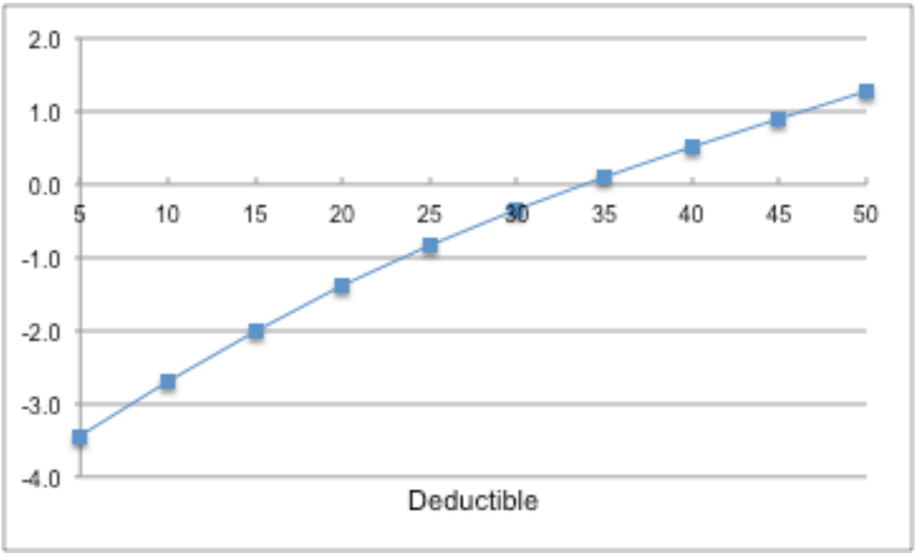

FIGURE 3 - INSURER'S NET PROFIT FOR CASE 2

Figures 3 shows that insurer's net profits are positive starting from deductible 35.

Regarding the insurer's adjustment coefficient, it increases when the deductible does. As Figure 4 shows, the maximum insurer's adjustment coefficient is 50. Consequently, the optimal layer is $100\ xs\ 50$.

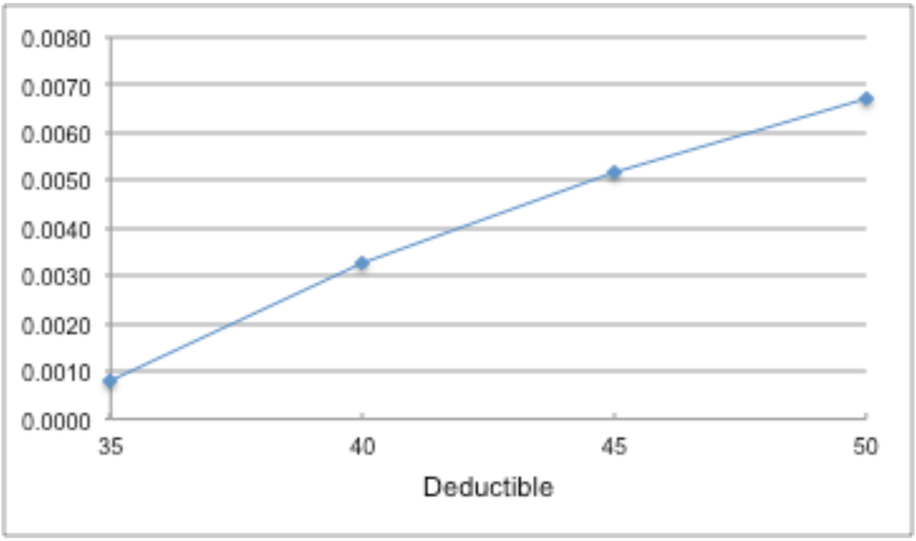

FIGURE 4 - INSURER'S ADJUSTMENT COEFFICIENT FOR CASE 2

The importance to find the optimal layer, i.e. the layer that maximize the value of the insurer's adjustment coefficient, is because this value minimizes the upper bound for the probability of ultimate ruin.

We found, through this example, that for an excess of loss reinsurance with reinstatements, with initial reinsurance premium calculated under the expected value principle and aggregate claim of portfolio distributed as a compound Poisson, the insurer's adjustment coefficient is an unimodal function of the deductible (l).

This result is in accordance to the one proved by Waters (1983) when an excess of loss reinsurance contract is considered.

9. Conclusions

In this thesis we studied how to calculate de initial reinsurance premium for excess of loss reinsurance contracts with aggregate layer under different premium principles such as pure premium, expected value premium, standard deviation premium and the proportional hazard premium principles.

Moreover, we presented a general method for calculating the adjustment coefficient of the retained risk for an excess of loss reinsurance with reinstatements when there is no aggregate deductible.
Our method considers constant percentages of the reinstatement premiums, regarding the premium initially paid, only to have more friendly formulae to compute the adjustment coefficient for the retained risk, but the results can be extended to include different percentages for each reinstatement using the formulae introduced in Chapter 3.

We found, for the particular case described in Chapter 8, that for an excess of loss reinsurance with reinstatements, with initial reinsurance premium calculated under the expected value principle and aggregate claim of portfolio distributed as a compound Poisson with truncated Pareto, the insurer's adjustment coefficient is a unimodal function of the deductible (l).

In order to have an accurate value of the optimal layer, a small step of discretisation should be used. We recommend the improvement of the algorithm in R for calculating the joint distribution using bivariate Panjer's recursion formula for any step of discretization h. Additionally, further research may be addressed considering the introduction of an aggregate deductible.

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