When Micro Prudence increases Macro Risk: The Destabilizing Effects of Financial Innovation, Leverage, and Diversification*

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Abstract

By exploiting basic common practice accounting and risk management rules, we propose a simple analytical dynamical model to investigate the effects of micro-prudential changes on macro-prudential outcomes. Specifically, we study the consequence of the introduction of a financial innovation that allow reducing the cost of portfolio diversification in a financial system populated by financial institutions having capital requirements in the form of VaR constraint and following standard mark-to-market and risk management rules. We provide a full analytical quantification of the multivariate feedback effects between investment prices and bank behavior induced by portfolio rebalancing in presence of asset illiquidity and show how changes in the constraints of the bank portfolio optimization endogenously drive the dynamics of the balance sheet aggregate of financial institutions and, thereby, the availability of bank liquidity to the economic system and systemic risk. The model shows that when financial innovation reduces the cost of diversification below a given threshold, the strength (due to higher leverage) and coordination (due to similarity of bank portfolios) of feedback effects increase, triggering a transition from a stationary dynamics of price returns to a non stationary one characterized by steep growths (bubbles) and plunges (bursts) of market prices.

JEL classification: E51, G11, G18, G21.

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1 Introduction

In most standard economic models, financial institutions are viewed as passive players and credit does not have any macroeconomic effect. Yet, a growing body of empirical literature consistently finds that an acceleration of credit growth is the single best predictor of future financial instability (see Schularick and Taylor 2012, Gourinchas, Valdes and Landerretche 2001; Mendoza and Terrones 2008; Borio and Drehmann 2009). These empirical results confirm that the balance sheet dynamics of financial intermediaries, far from being passive and exogenous, is instead the "endogenous engine" that drives the boom-bust cycles of funding and liquidity and hence the dynamics of systemic risk. As stated by Adrian and Shin (2010): "balance sheet aggregates such as total assets and leverage are the relevant financial intermediary variables to incorporate into macroeconomic analysis". In fact, a change in the total assets of the financial institutions has important consequences in driving the financial cycles through their influence on the availability of credit and funding of real activities. In this way changes in the total asset and leverage of financial intermediaries play a key role in determining the level of real activity. However, while the proximate cause for crises is very often an expansion of the balance sheets of financial intermediaries, the reasons for the acceleration of credit growth remain unclear.

In this paper we investigate the determinants of the balance sheet dynamics of financial intermediaries by modeling the dynamic interaction between asset prices and bank behavior induced by regulatory constraints. We find that standard capital requirements, in the form of Value–at–Risk (VaR) constraints, together with the level of diversification costs (related to the availability of derivatives products), determine bank decisions on diversification and leverage which, in turns, strongly affect the dynamics of traded assets through the bank strategies of portfolio rebalances in presence of a finite asset liquidity. We can then show how changes in the constraints of the bank portfolio optimization (such as changes in the prevailing cost of diversification or changes in the micro-prudential policies) endogenously drive the dynamics of systemic risk and the availability of bank liquidity to the economic system.

Therefore, our model provides a simple analytical framework that can be used to investigate questions such as: (i) what is the effect of financial innovations allowing a more efficient diversification of risk? (ii) what are the systemic risk consequences of a change in capital requirements and in the maximum leverage allowed to banks? (iii) How does contagion propagates when financial players change their level of diversification? (iv) Is it always beneficial for the system to have more diversified portfolios? (v) Can actions to reduce risk of a single financial institution increase the risk of a systemic event i.e. are micro-prudential policies always coherent with macro-prudential objectives?

In building our model we try to keep behavioral assumptions at minimum, exploiting instead the implications of "objective" constraints imposed by regulatory institutions and standard market practice. We then start from a simple portfolio optimization problem in presence of cost of diversification and VaR constraint[1] showing how a reduction in the costs of diversification (due, for instance, to financial innovations such as securization) leads to an increase in both leverage and diversification and how a positive relation naturally exists between these two latter variables: an increase in diversification, by reducing the portfolio volatility, relaxes the VaR constraint allowing to increase the bank leverage.

So a first important result is that financial innovation which, by increasing the optimal level of diversification, reduces idiosyncratic risks, actually increases the exposure to undiversifiable macro risks by increasing the optimal leverage of a VaR constrained investor. Moreover, a higher level of diversification, by increasing the overlap among bank portfolios, increases the correlation among them. Thus, the combined increase in risk exposure and correlation of financial institutions will expose the economy to higher level of systemic risk.

We then link these results with the literature on the portfolio rebalancing induced by the mark-to-market accounting rules and VaR constraint (see for instance Adrian and Shin 2009). In this balance sheet models an increase in the value of the assets, increases the amount of equity leading to surplus of capital with respect to the VaR requirements which is adjusted by expanding the asset side through borrowing i.e. by raising new debt (typically done with repos contracts). Hence, VaR capital requirements, induce a perverse demand function: financial institution will buy more assets if their price rises and (with an analogous

[1]Note that VaR type of constraints arise from the capital requirements contained in Basel I and II bank regulations but also from margin on collateralized borrowing imposed by creditors (see Brunnermeier and Pedersen 2008), rating agencies, and internal risk management models.

mechanism but with reversed sign) sell more assets when their price falls. Therefore, a VaR constrained financial institution will have positive feedback effect on the prices of the assets in his portfolio.[2] The intensity and coordination (among financial institutions) of these portfolio rebalancing feedbacks will depend, respectively, on the degree of leverage and diversification.

By analyzing the endogenous asset price dynamics determined by the impact of supply and demand generated by the financial institutions rebalancing their portfolio, we derive closed–form expression for the endogenous component of the variance and covariance of asset price and bank portfolios. We can then show that a larger leverage increases both the variances and the covariances of stock returns while a greater degree of diversification reduces the variances but increases the covariances. Both are positively related with correlations. Through these explicit formula linking portfolio choices and the statistical properties of asset price dynamics, we can investigate the impact of changes in bank costs and constraints (micro–prudential policy) on systemic risk and on the dynamics of the banking sector total asset (macro–prudential objectives).

Our paper tries to combine several strands of literature: (i) the one on the impact of the imposition of capital requirements on the behavior of financial institutions and their possible procyclical effects (Danıelsson et al., 2004; Danielsson et al., 2009; Adrian and Shin, 2009; Adrian et al., 2011; Adrian and Boyarchenko, 2012; Tasca and Battiston, 2012); (ii) the literature on the effects of diversification and overlapping portfolios on systemic risk (Tasca and Battiston, 2011; Caccioli et al., 2012); (iii) the literature on the risks of financial innovation (Brock et al., 2009; Caccioli et al., 2009; Haldane and May, 2011); (iv) the literature on distressed selling and its impact on the market price dynamics (Kyle and Xiong, 2001; Wagner, 2011; Cont and Wagalath, 2011; Thurner et al., 2012; Cont and Wagalath, 2012; Caccioli et al., 2012); (v) the literature on the determinants of the dynamics of balance sheet aggregates and credit supply of financial institutions (Stein 1998, Bernanke and Gertler 1989, Bernanke, Gertler and Gilchrist 1996, 1999 and Kiyotaki and Moore 1997). Our contribution is to propose a simple model that, by combining these different streams

[2]This type of active balance sheet management is particularly utilized by investment banks, ABS issuers, security broker-dealers, i.e. by the so called market-based financial intermediaries or shadow banking system.

of literature, provides a fully analytical quantification of the links between micro prudential rules and macro prudential outcomes in a multivariate context which considers both the presence of endogenous feedback caused by portfolio rebalancing and the impact of financial innovations on the cost of diversification.

The paper is organized as follows. Section 2 presents the model set up and the analytical results by first describing the portfolio decision problem of financial institutions facing VaR constraints and diversification costs and then analyzing its macroeconomic consequences in the dynamic case which considers the impact of investor demands on the asset dynamics. Section 3 analyzes the systemic risk implications of our model both a static setting without feedback and in a dynamic setting with the endogenous feedback generated by portfolio rebalancing. Based on those analytical results, Section 4 discusses the macro-prudential consequences of the introduction of financial innovations reducing diversification costs. Section 5 summarizes and concludes.

2 The model

2.1 Portfolio decisions

We begin by considering a financial institution endowed with a given amount of initial equity capital E and we model its portfolio selection across a collection of risky investments $i = 1, ..., M$. In general, these might be individual investments or asset classes. Financial institutions, correctly perceive that each risky investment entails both an idiosyncratic (diversifiable) risk component and a systematic (undiversifiable) risk component, i.e. the perceived variance of the investment risky asset i, σ_i^2, can be decomposed as $\sigma_i^2 = \sigma_s^2 + \sigma_d^2$ where σ_s^2 is the systematic risk and σ_d^2 is the diversifiable risk component. Hence, the expected mean and volatility per dollar invested in the portfolio chosen by a given institution are μ and $\sigma_p = \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}$, respectively. For simplicity, we assume that financial institutions adopt a simple investment strategy consisting in forming an equally weighted portfolio composed of m randomly selected risky investments from the whole collection of M available investment assets.

Because of the presence of transaction costs, firms specialization and other type of frictions, we assume the existence of "costs of diversification" which, in general, can prevent each institution to achieve full diversification of its portfolio (precisely the existence of these costs in real markets spurred the developments of financial innovation products as we will discuss in the next sections).

Let r_L be the per dollar average interest expense on the liability side, then the Net Interest Margin (NIM) of the financial institution is $\mu - r_L$. The NIM is therefore a measure of the overall profitability of a financial institution.

We then assume that financial institutions seek to maximize returns from the risky investments under their Value at Risk (VaR) constraints. The VaR is some multiple of the standard deviation of the portfolio of assets A. With σ_p the holding period volatility per dollar of asset A and α a scaling constant, the VaR constraint faced by the financial institution is

$$VaR = \alpha \sigma_p A \leq E. \tag{1}$$

As empirically shown by Adrian and Shin (2009) financial institutions adjust their asset side rather than raising or redistributing equity capital. In agreement with these empirical observations, we will consider the equity capital of the financial institutions to be fixed.

Summarizing, given their NIM and level of equity E, financial institutions, facing cost of diversification and VaR constraints, choose the level of total asset A and degree of diversification m which maximize their returns from the risky investments. That is, assuming cost of diversification proportional to m, financial institutions maximize

$$\max_{A,m} \quad A(\mu - r_L) - \tilde{c}m \qquad \text{s.t.} \quad \alpha A \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq E. \tag{2}$$

where $\tilde{c}$ is the cost for investment (assumed to be the same across all investments). Dividing by E and defining $c = \tilde{c}/E$, we can express the maximization problem in terms of the leverage $\lambda = \frac{A}{E}$,

$$\max_{\lambda,m} \quad \lambda(\mu - r_L) - cm \qquad \text{s.t.} \quad \alpha \lambda \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq 1. \tag{3}$$

Hence, each institution chooses the optimal leverage $\lambda^* = A^*/E$ and the optimal number of investments m^* which maximizes its Return On Equity (ROE) under its VaR constraints.

It is convenient to transform the constraint by squaring both sides so that the Lagrangian can be written as

$$L = \lambda(\mu - r_L) - cm - \frac{1}{2}\gamma\left(\alpha^2\lambda^2\left(\sigma_s^2 + \frac{\sigma_d^2}{m}\right) - 1\right). \tag{4}$$

where γ is the Lagrange multiplier for the VaR constraint. The first order condition for λ is

$$(\mu - r_L) - \gamma\alpha^2\sigma_p^2\lambda = 0 \qquad \Rightarrow \qquad \lambda = \frac{1}{\gamma}\frac{1}{\alpha^2}\frac{\mu - r_L}{\left(\sigma_s^2 + \frac{\sigma_d^2}{m}\right)} \tag{5}$$

Substituting in the constraint we obtain the Lagrange multiplier or shadow price of the VaR constraint γ

$$\gamma = \frac{1}{\alpha}\frac{\mu - r_L}{\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}} = \frac{1}{\alpha}\frac{\mu - r_L}{\sigma_p} \tag{6}$$

which is proportional to the Sharpe ratio. The optimal number of investments m^* is then,

$$m^* = \frac{\sqrt{\gamma}\alpha\lambda\sigma_d}{\sqrt{2c}} = \lambda\sigma_d\sqrt{\frac{\alpha}{2c}\frac{\mu - r_L}{\sigma_p}} \tag{7}$$

which shows that, as expected, the level of diversification chosen is inversely related to the cost of diversification c. For the leverage we have,

$$\lambda^* = \frac{1}{\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}} = \frac{1}{\alpha\sigma_p} \tag{8}$$

thus, the optimal leverage is inversely related to the volatility of the asset portfolio. In the following, we will drop the star symbol on the optimal values for notational convenience, i.e. we will denote the target leverage λ^* and diversification m^* simply as λ and m, respectively.

Assuming that all financial institutions have the same perceived volatility and face the same cost of diversification c, the portfolio decision becomes the same for all institutions so that the degree of diversification m and leverage λ will be the same across banks.

Figure 1 reports the numerical solutions for the optimal leverage as a function of different levels of diversification costs (and for a given choice of the set of the remaining parameters in the model). Each line corresponds to different levels of systematic to idiosyncratic noise ratio ($\sigma_s/\sigma_d = \{0,\ 0.3,\ 0.6\}$). A reduction of diversification costs, by increasing the level of diversification and hence relaxing the VaR constraint, allows the financial institution to

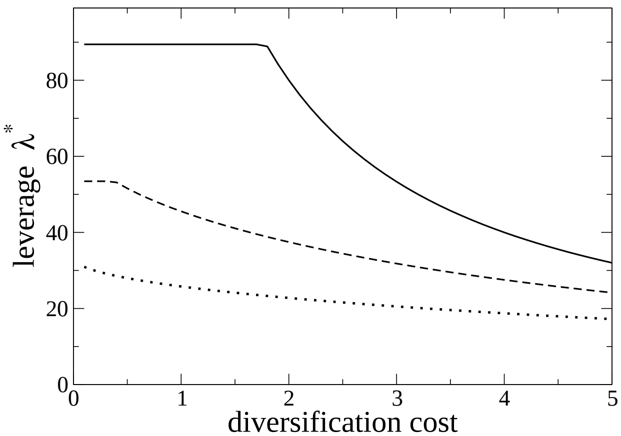


Figure 1: Relation between the optimal leverage λ and the diversification cost c, obtained by solving numerically Eq.s (7) and (8). The used parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).

increase the optimal leverage, especially for lower level of the systematic to idiosyncratic noise ratio. Note that below a given cost the optimal leverage becomes constant due to the saturation of diversification reached when the portfolio becomes perfectly diversified across all the M available investments. The sensitivity of the optimal leverage to diversification costs is higher for lower systematic to idiosyncratic noise ratios.

2.2 Overlapping portfolios

We now assume that our economy is composed by a group of N financial institutions labeled with $j = 1, ..., N$ and investing in the M risky investments as described above. The portfolio holdings of the N banks can be represented by using a bipartite graph, where the first set of nodes is composed by the N banks and the second set of nodes is composed by the M risky investments. Each bank j invests in m_j investments and this fact can be represented by m_j links connecting node j with its m_j investments.

In the following of the paper we will make the simplifying assumption of homogeneity across banks, i.e. a sort of "representative bank" hypothesis. This assumption allows us to solve analytically the model. As said above, homogeneity means that all the banks have the same equity E and thus each bank solves the same optimization problem and invests in the same number m of risky investments. However we will assume that each bank chooses randomly and independently the m investments across the set of the M available ones and thus the portfolios are different for different banks. A realization of portfolio choice of all the banks lead to a specific instance of the bipartite graph.

The number of banks n having a specific risky investment in their portfolio is a random variable described by the binomial distribution

$$P(n; N, M, m) = \binom{N}{n} \left(\frac{m}{M}\right)^n \left(1 - \frac{m}{M}\right)^{N-n} \tag{9}$$

whose mean value is clearly $E[n] = mN/M$.

Taken two banks, we can define the overlap o of their portfolios as the number of risky investments in common in the two portfolios. Also o is a random variable and it is distributed as an hypergeometric distribution

$$P(o; M, m) = \frac{\binom{m}{o}\binom{M-m}{m-o}}{\binom{M}{m}} \qquad 0 \leq o \leq m. \tag{10}$$

Its mean value is $E[o] = m^2/M$ and its variance is $V[o] = m(M-m)^2/(M^2 - M)$. Finally, the fractional overlap of two portfolios $o_f = o/m$ is a number between 0 and 1 describing which fraction of the portfolio is in common between the two banks. Clearly, the mean fractional overlap is $\bar{o} \equiv E[o_f] = m/M$, therefore the value of the portfolio size m is also a measure of the average fractional overlap $\bar{o}$ between portfolios and viceversa.

The left panel of Figure 2 shows the numerical solutions of the fractional overlap, coming from the optimal portfolio decision, as a function of different levels of diversification costs (again, each line corresponds to different levels of systematic to idiosyncratic noise ratio, $\sigma_s/\sigma_d = \{0,\ 0.3,\ 0.6\}$). The figure shows how reducing the costs of diversification, by the introduction of some new form of financial products for example, increases the degree of overlap and hence correlation, between the portfolio of financial institutions.

The fractional overlap resulting from the portfolio choices of financial institutions, can also be represented as a function of the tightness of the imposed capital requirements. This relation, depicted in the right panel of Figure 2, implies that regulator could tune the required

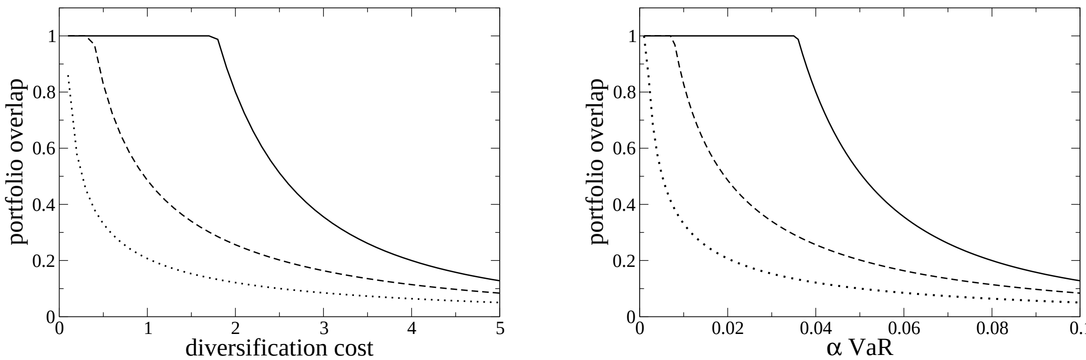


Figure 2: The left panel shows the mean fractional overlap $\bar{o}$ between two portfolios versus the diversification cost c and the right panel shows $\bar{o}$ versus the α parameter of the VaR constraint. The used parameters are: $M = 20$, $\mu - r_L = 0.8$, $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). Moreover in the left panel we set $\alpha = 0.05$ and in the right panel we set $c = 2.5$.

capital ratio α so to reach a given level of overlap, and hence correlation, among financial institutions.

2.3 Asset demand from portfolio rebalancing

Having identified their optimal leverage, financial institution periodically rebalance their portfolios in order to maintain the desired target leverage. The rebalancing of the portfolio of individual bank j at time t, is given by the difference between the desired amount of asset $A^*_{j,t} = \lambda E_{j,t}$ and the actual one $A_{j,t}$,[3] i.e. $\Delta R_{j,t} \equiv A^*_{j,t} - A_{j,t}$. By defining the realized return portfolio $r^p_{j,t}$, $\Delta R_{j,t}$ can be written as (see Appendix A)

$$\Delta R_{j,t} = (\lambda - 1) r^p_{j,t} A^*_{j,t-1}, \tag{11}$$

that is, any profit or loss from investments in the chosen portfolio ($r^p_{j,t} A^*_{j,t-1}$) will directly result in a change in the asset value amplified by the current degree of leverage (being

[3]As clearly shown by Adrian and Shin (2009), the balance sheet adjustments are typically performed by expanding or contracting the asset side rather than the level of equity.

$\lambda > 1$). Hence, a VaR constrained financial institution will have a positive feedback effect on the prices of the assets in his portfolio.

The total demand of the risky investment i at time t will be simply the sum of the individual demand of the financial institutions who picked investment i in their portfolio,

$$D_{i,t} = \sum_{j=1}^{N} I_{\{i\in j\}} \frac{1}{m} \Delta R_{j,t} = \sum_{j=1}^{N} I_{\{i\in j\}} (\lambda - 1) r^p_{j,t} \frac{A^*_{j,t-1}}{m} \tag{12}$$

where $I_{\{i\in j\}}$ is the indicator function which takes value one when investment i is in the portfolio of institution j and zero otherwise.

By considering total assets approximately the same across financial institutions, $A^*_{j,t-1} \simeq A^*_{t-1}$, the demand of the investment i can be approximately rewritten in terms of individual investments returns $r_{i,t}$ as (see Appendix B.1):

$$D_{i,t} \approx (\lambda - 1) \frac{A^*_{t-1}}{m} \frac{N}{M} \left(r_{i,t} + \frac{m-1}{M-1} \sum_{k\neq i} r_{k,t} \right) \tag{13}$$

From Equation (13) we can notice that, in the hypothetical case in which all investment returns are approximately the same, the demand of an investment i does not depends on the degree of diversification m. This is due to a compensation happening between two opposite effects: on one hand, increasing the diversification of the investor increases the average number of investors ($\frac{Nm}{M}$) holding the investment i in their portfolio, increasing the expected demand of investment i; on the other hand, it reduces the share of the total asset rebalancing ($\frac{A^*_{t-1}}{m}$) borne by the investment i, thus reducing the impact on the demand of i. These two effects turn out to exactly offset each other on expectation.

Although m does not affect the expected demand, it does heavily change the variance, and most importantly, the correlation between the demand of two different investments. In fact, by using (13), we can compute the variance of the demand, $Var[D_i]$, and the demand correlation between two investment assets, $\rho(D_i, D_j)$ (see Appendix B.2). Figure 3 shows the variance and correlation of demand as a function of the mean fractional overlap between portfolios (again for different levels of systematic to idiosyncratic noise ratios). While the variance of the demand of a given asset decreases with the portfolio overlap, cross correlation quickly increases with the portfolio overlap and tends to one as the portfolio becomes perfectly diversified. In fact, as shown in the appendix, $\rho(D_i, D_j) \underset{m\to M}{\to} 1$. Clearly, correlations are in general stronger for higher level of volatility coming from the systematic component.

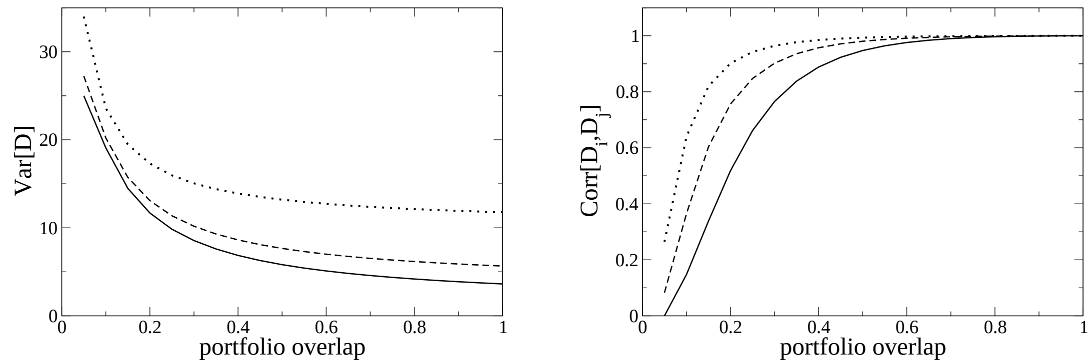


Figure 3: Variance (left) and cross correlation (right) of demand as a function of the mean fractional overlap $\bar{o}$ between portfolios. The parameters are $M = 20$, $N = 100$, and $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line).

2.4 Risky asset dynamics with endogenous feedbacks

In this section we study the dynamics of the model in the case where the return of the risky investments are endogenously influenced by the former period demands coming from the portfolio rebalancing of financial institutions. In presence of rebalancing feedbacks, the return process will now be made of two components:

$$r_{i,t} = e_{i,t-1} + \varepsilon_{i,t} \tag{14}$$

the exogenous component $\varepsilon_{i,t}$ coming from the external shocks and the endogenous component $e_{i,t-1}$ coming from the previous period portfolio rebalancing of the financial institutions.

We assume that the exogenous component has a multivariate factor structure

$$\varepsilon_{i,t} = f_t + \epsilon_{i,t}, \tag{15}$$

with the factor f_t and the idiosyncratic noise $\epsilon_{i,t}$ uncorrelated and distributed with mean zero and constant volatility, respectively σ_f and σ_ϵ (the same for all investments). Thus, the variance of the exogenous component of the risky investment i is $V(\varepsilon_i) = \sigma_f^2 + \sigma_\epsilon^2$.

Assuming, for simplicity, a linear price impact function, the endogenous component of the return of investment i at time $t+1$ becomes[4]

$$e_{i,t} = \frac{D_{i,t}}{\gamma_i C_{i,t}} \tag{16}$$

where $C_{i,t} = \sum_{j=1}^{N} I_{\{i\in j\}} \frac{A^*_{j,t-1}}{m}$ is a proxy for market capitalization of investment i, and γ_i is a parameter expressing the market liquidity of the investment i. Again, assuming that $A^*_{j,t-1} \simeq A^*_{t-1}$, the market capitalization of investment i is

$$C_{i,t} = \sum_{j=1}^{N} I_{\{i\in j\}} \frac{A^*_{j,t-1}}{m} \approx \frac{A^*_{t-1}}{m} \sum_{j=1}^{N} I_{\{i\in j\}} = \frac{N}{M} A^*_{t-1} \tag{17}$$

because on average there are Nm/M banks having investment i in their portfolio.

Substituting Equations (13), (14), and (17) in (16) and using matrix notation we obtain the following Vector Autoregressive (VAR) dynamics of the vector of the endogenous components

$$\boldsymbol{e_t} = \boldsymbol{\Phi}\, \boldsymbol{r_t} = \boldsymbol{\Phi}\,(\boldsymbol{e_{t-1}} + \boldsymbol{\varepsilon_t}) \tag{18}$$

where $\boldsymbol{\Phi} \equiv (\lambda - 1)\;\; \boldsymbol{\Gamma}^{-1}\;\; \boldsymbol{\Psi}$ with

$$\underset{M\times M}{\boldsymbol{\Gamma}} = \begin{bmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_M \end{bmatrix}, \qquad \underset{M\times M}{\boldsymbol{\Psi}} = \begin{bmatrix} \frac{1}{m} & \frac{1}{m}\frac{m-1}{M-1} & \dots & \frac{1}{m}\frac{m-1}{M-1} \\ \frac{1}{m}\frac{m-1}{M-1} & \frac{1}{m} & \dots & \frac{1}{m}\frac{m-1}{M-1} \\ \vdots & & \ddots & \vdots \\ \frac{1}{m}\frac{m-1}{M-1} & \frac{1}{m}\frac{m-1}{M-1} & \dots & \frac{1}{m} \end{bmatrix}.$$

Thus the endogenously determined component of returns follows a VAR process of order one. The dynamics of such VAR(1) process is dictated by the eigenvalues of the matrix $\boldsymbol{\Phi} = (\lambda - 1)\boldsymbol{\Gamma}^{-1}\boldsymbol{\Psi}$. In particular, since the maximum eigenvalue of the matrix $\boldsymbol{\Psi}$ is always equal to 1 (see Appendix C), the maximum eigenvalue of the VAR(1) process becomes

$$\Lambda_{\max} \approx (\lambda - 1)\;\overline{\gamma^{-1}} \tag{19}$$

where $\overline{\gamma^{-1}}$ is the average of all the γ_i^{-1}. Hence, the maximum eigenvalue depends on the degree of leverage and on the average illiquidity of the investments.

[4]A stochastic component coming from the exogenous demands of traders not actively rebalancing their portfolio could be added at the cost of complicating the subsequent computations.

When the maximum eigenvalue is greater than one, the return processes become non-stationary and explosively accelerating. It is important to remark that even a reduction in the liquidity of only one risky investment (by changing the average illiquidity of the investments) impacts the dynamics of all the traded investments and can potentially drive the whole financial system towards instability. In fact, depending on the average of the $\frac{1}{\gamma_i}$, the maximum eigenvalue (and thus the dynamical properties of the whole system) will be highly sensitive to illiquid investments, i.e. to investment having a small γ. For the sake of simplicity and analytical tractability, in the rest of the paper we will consider only the case in which all investments have the same liquidity, i.e. $\gamma_i = \gamma$, $\forall i$.

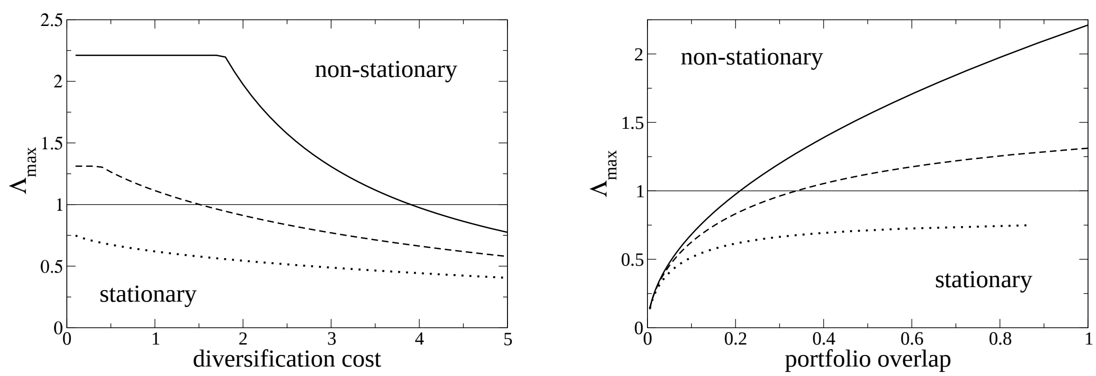


Figure 4: The left panel shows the maximum eigenvalue $\Lambda_{\max}$ as a function of the diversification cost, while the right panel shows $\Lambda_{\max}$ as a function of the mean fractional overlap $\bar{o}$ between two portfolios versus. The used parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, and $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). The horizontal solid line shows the condition $\Lambda_{\max} = 1$, therefore the return dynamics is stationary below this line and non stationary above it.

Figure 4 shows the maximum eigenvalue $\Lambda_{\max}$ as a function of the diversification cost c (left panel) and as a function of the mean portfolio overlap $\bar{o}$ (right panel). We notice that a reduction of the diversification cost tends to reinforce the feedback induced by portfolio rebalancing which can lead to dynamic instability of the system (for $\Lambda_{\max} > 1$) when the diversification costs decrease below a certain threshold (which is higher for smaller ratio of systematic to idiosyncratic volatility). Analogously, we can analyze the dependence of the maximum eigenvalue of the dynamical system from the degree of portfolio overlap among the financial institutions. A higher level of coordination in portfolio rebalancing, due to similarities in the portfolio compositions, also reinforces the aggregate feedback between market prices and balance sheet values pushing the system toward the region of instability $\Lambda_{\max} > 1$. Note however, that the transition to a non stationary process is not achieved when the portfolio overlap is equal to one, but, depending on the other parameters, also a moderate value of the portfolio overlap can lead to market instability.

We now show that the homogeneity assumptions allow us to give an exact description of the dynamics of investment returns and to compute in closed form the variance-covariance matrix of returns. In fact, notice that $m\boldsymbol{\Psi}$ can be written as

$$m\boldsymbol{\Psi} = (1-b)\mathbf{I} + b\boldsymbol{\iota}\boldsymbol{\iota}' \tag{20}$$

with the scalar $b = \frac{m-1}{M-1}$, identity matrix $\mathbf{I}$, and the column vector of ones $\boldsymbol{\iota}$. Hence, the VAR for the vector of endogenous components in equation (18) can be rewritten as

$$\boldsymbol{e_t} = (1-b)\,\mathcal{A}(\boldsymbol{e_{t-1}} + \boldsymbol{\varepsilon_t}) + b\,M\mathcal{A}\boldsymbol{\iota}(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{21}$$

with matrix $\mathcal{A} \equiv \frac{\lambda-1}{m}\boldsymbol{\Gamma}^{-1}$ and scalars $\bar{e}_t \equiv \frac{1}{M}\sum_{k=1}^{M} e_{k,t}$ and $\bar{\varepsilon}_t \equiv \frac{1}{M}\sum_{k=1}^{M} \varepsilon_{k,t}$. The scalar $\bar{e}_t$ can be interpreted as the endogenous return of the market portfolio. Thus, the endogenous component of an individual investment becomes

$$e_{i,t} = (1-b)\,a_i(e_{i,t-1} + \varepsilon_{i,t}) + b\,Ma_i(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{22}$$

with scalar $a_i = \frac{\lambda-1}{m\gamma_i}$.

Therefore, the process for $e_{i,t}$ can be rewritten as a linear combination of a standard univariate AR(1) process and a dynamic process depending on the averages of previous period endogenous components and shocks. In this way, $e_{i,t}$ is a mixture of a perfectly idiosyncratic process (i.e. uncorrelated with the others investment processes) receiving weight $1-b$ and a perfectly correlated process with weight b. Being $b = \frac{m-1}{M-1}$, the higher is the value of m, the higher is the weight given to the perfectly correlated component of mixture and, hence, the higher the correlations among the endogenous components of the different investments.

Moreover, assuming $a_i = a \;\forall i$ (i.e. all investments have the same liquidity), the process for $\bar{e}_t$ becomes:

$$\bar{e}_t \;=\; a(1-b+bM)(\bar{e}_{t-1} + \bar{\varepsilon}_t) \equiv \phi\,(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{23}$$

with $\phi \equiv a(1 - b + bM)$. Therefore, the dynamics of the average process $\bar{e}_t$ is also an autoregressive of order one; its variance, assuming stationarity of e_t, is (see Appendix D)

$$V(\bar{e}_t) = \frac{\Lambda_{\max}^2}{1 - \Lambda_{\max}^2}\, V(\bar{\varepsilon}_t) \tag{24}$$

with $V(\bar{\varepsilon}_t) = \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}\right)$.

Finally, defining the distance of the endogenous component of investment i from the average as $\Delta e_{i,t} \equiv e_{i,t} - \bar{e}_t$, we also have that

$$\Delta e_{i,t} = (1 - b)\, a(\Delta e_{i,t-1} + \Delta \varepsilon_{i,t}) \tag{25}$$

where $\Delta\varepsilon_{i,t} \equiv \varepsilon_{i,t} - \bar{\varepsilon}_t$. So that the dynamics of the individual distance of the endogenous component of investment i from the average value $\bar{e}_t$ is also an autoregressive process of order one.

We can then interpret the dynamics of the endogenous components of each individual investment as an idiosyncratic AR(1) process around a common process for the average value also following an AR(1) and where the amplitude of the idiosyncratic component is inversely related to the portfolio diversification. In other words, the dynamics of endogenous returns can be described as a multivariate "ARs around AR". When the process is stationary, the mean market behavior is described by a mean reverting process. In turn, each investment performs a mean reverting process around the market mean. It can be shown that the time scale for mean reversion of the market is always larger than the time scale of reversion of an investment toward the market mean behavior. Moreover, when m increases the time scale of reversion of individual investment declines, which means that investments become more quickly synchronized with the mean market behavior. Finally, notice that this type of multivariate "ARs around AR" dynamics is also followed by the endogenous component of portfolio returns $e_t^p \equiv \frac{1}{m}\sum_{k=1}^{m} e_{k,t}$ where the number of assets in portfolio is $m < M$.

Importantly, this representation clearly shows that, as for the exogenous component, also the variability of the endogenous component of returns can be decomposed into a systematic component associated with the volatility of $\bar{e}_t$ and an idiosyncratic one. Therefore, both the exogenous and endogenous components contain a diversifiable and undiversifiable source of risk, so that also the total risk of the investments return is composed of these two type of risk, σ_s and σ_d, as perceived by the financial institutions.

Thanks to this representation we are able to explicitly compute the variance and covariances of the process for the endogenous components $e_{i,t}$, which are reported in Appendix (D). It can be shown that a larger leverage increases both the variances and the covariances of $e_{i,t}$, while a greater degree of diversification reduces the variances and increases the covariances. Both are positively related with correlations. In particular, it can be shown (see Appendix D) that the correlations among the endogenous returns tend to one as $m \to M$.[5]

Taking into account the feedback induced by the portfolio rebalancing introduces a new endogenous component in the variance of the investment asset given by the variance of the endogenous component

$$V(r_{i,t}) = V(e_{i,t}) + V(\varepsilon_{i,t}) \tag{26}$$

where the exogenous variance $V(\varepsilon_{i,t}) = \sigma_f^2 + \sigma_\epsilon^2$ and the explicit expression for the endogenous variance $V(e_{i,t})$ is given in Appendix D. This expression shows that the endogenous component of return leads to an increase of the volatility of an investment above its "bare" level $V(\varepsilon_{i,t})$. This volatility increase is at the end due to the finite liquidity of the investments and disappears when $\gamma \to \infty$ and it can therefore be seen as an "illiquidity induced contribution to volatility".

Analogously, the covariance between the returns of two investments is enhanced by the contribution coming from the covariance between the endogenous components (see Appendix D)

$$Cov(r_{i,t}, r_{j,t}) = Cov(e_{i,t}, e_{j,t}) + \sigma_f^2. \tag{27}$$

Figure 5 shows the variance of returns and the correlation between the endogenous component of returns of two investments as a function of diversification cost c. We see that when cost is high, variance and correlations are low. By decreasing cost, variance of returns increases as well as correlations. If the market factor is not strong enough, there is a value of c for which variance diverges, corresponding to the case where the maximum eigenvalue $\Lambda_{\max}$ becomes equal to one. In this limit, correlations become closer and closer to one.

[5] Notice that the endogenous correlations would not tend to one in presence of an additional stochastic component in the price impact function (Eq. 16) coming from the exogenous demand of traders not actively rebalancing their portfolio (see Footnote 4).

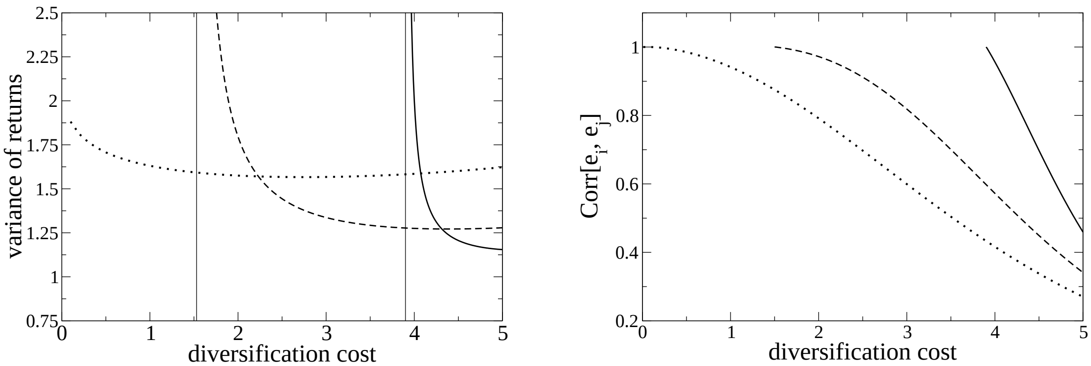


Figure 5: The left panel shows the variance of investment returns, $V(r_{i,t})$, of Eq. (26) as a function of the diversification cost c, while the right panel shows the correlation of the endogenous component of returns between two investments, $Corr(e_{i,t}, e_{j,t})$ as a function of c. The used parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, and $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). The vertical lines in the left panel indicate where the variance of returns diverges and below these values correlations in the right panel are clearly not defined.

As a consequence the variance of portfolio returns in presence of the rebalancing feedbacks becomes

$$\begin{aligned} V(r_t^p) &= \frac{V(e_{i,t})}{m} + \frac{m-1}{m} Cov(e_{i,t}, e_{j,t}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m} \\ &= V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}, \end{aligned} \tag{28}$$

which, as for investment returns, means that the endogenous component (and therefore the illiquidity of the assets) increases the volatility of portfolio by a illiquidity induced contribution. The left panel of Figure 6 shows the variance of portfolios as a function of diversification cost c for different values of the ratio σ_s/σ_d. It is important to notice that by reducing diversification cost, the variance of portfolios initially declines. This means that in this regime, financial innovation makes portfolios less risky and it is therefore beneficial. However, the variance of the portfolio reaches a minimum for a given value of c and by reducing further the diversification cost, one gets closer and closer to the critical condition $\Lambda_{\max} = 1$ and the variance increases without bounds. In this regime, even small variations of the cost lead to huge increases of the riskiness of the portfolios.

It is also interesting to note that the variance of the observed market portfolio (the one containing all the M investments with equal weights) is

$$\begin{aligned} V(r_t^M) &= V(\bar{e}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M} \equiv V(\bar{e}) + \sigma_{p_M}^2 = \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2}\,\sigma_{p_M}^2 + \sigma_{p_M}^2 \\ &= \frac{1}{1-\Lambda_{\max}^2}\,\sigma_{p_M}^2 \end{aligned} \tag{29}$$

with $\sigma_{p_M}^2 \equiv \sigma_f^2 + \frac{\sigma_\epsilon^2}{M}$ being the market portfolio return when feedback due to impact is not present, i.e. corresponds to the case of an infinitely liquid market. So, the factor $\frac{1}{1-\Lambda_{\max}^2}$ representing the magnification of the exogenous variance due to the portfolio rebalancing, can then be termed the "variance multiplier" of the endogenous component. Clearly, for larger values of the maximum eigenvalue of the VAR process, the variance multiplier will increase exploding for $\Lambda_{\max}^2 \to 1$.

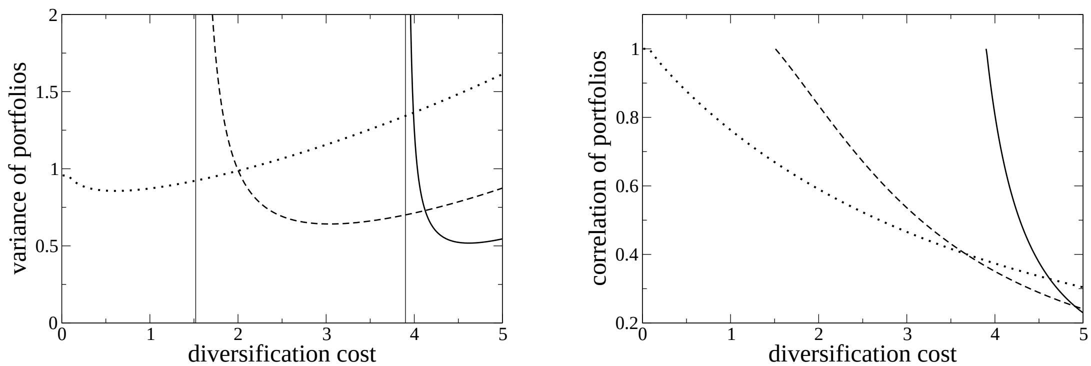


Figure 6: The left panel shows the variance of portfolios, $V(r_t^p)$, of Eq. (28) as a function of the diversification cost c, while the right panel shows their correlation, ρ_p^e, of Eq. (31). The used parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, and $\sigma_d = 1$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). The vertical lines in the right panel indicate where the variance of portfolios diverges and below these values correlations in the right panel are clearly not defined.

Similarly, the covariance between two portfolios containing m assets becomes

$$
\begin{aligned}
Cov(r^p_{h,t}, r^p_{k,t}) &= Cov(e_{i,t}, e_{j,t}) + \sigma_f^2 + m\frac{m}{M}\left(\frac{V(e_{i,t}) - Cov(e_{i,t}, e_{j,t}) + \sigma_\epsilon^2}{m^2}\right) \\
&= \frac{V(e_{i,t})}{M} + \frac{M-1}{M} Cov(e_{i,t}, e_{j,t}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M} \\
&= V(\bar{e}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M} \\
&= \frac{1}{1-\Lambda^2_{\max}}\,\sigma^2_{p_M}. \qquad (30)
\end{aligned}
$$

In fact, given the factor structure of $e_{i,t}$, with the factor being $\bar{r}_t = \bar{e}_{t-1} + \bar{\varepsilon}_t$, the covariance $Cov(e_{p,t}, \bar{e}_t)$ is equal to the variance of the factor $V(\bar{e})$ (as for the exogenous covariance).

Finally, the correlation between two portfolios in presence of endogenous feedback can be written as

$$
\begin{aligned}
\rho_p &= \frac{V(\bar{e}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}} = \frac{V(e_p)\frac{V(\bar{e})}{V(e_p)} + \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}\right)\frac{\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}}}{V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}} \\
&= \frac{\sigma_e^2 \rho_e + \sigma_\varepsilon^2 \rho_\varepsilon}{\sigma_e^2 + \sigma_\varepsilon^2} \qquad (31)
\end{aligned}
$$

where $\sigma_e^2 \equiv V(e_p)$, $\sigma_\varepsilon^2 \equiv \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}$, $\rho_e \equiv \frac{Cov(e_{p,t}, \bar{e}_t)}{V(e_p)} = \frac{V(\bar{e})}{V(e_p)}$, and $\rho_\varepsilon \equiv \frac{\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}}$. That is, the portfolio correlation in presence of active asset management is a weighted average of the endogenous correlations between e_p and $\bar{e}$, i.e. ρ_e, and the correlation between the exogenous shocks, (i.e. ρ_ε), with weights the respective variances σ_e^2 and σ_ε^2. Since both the endogenous ρ_e and exogenous ρ_ε correlations tend to one as $m \to M$, also the total correlation of the portfolio ρ_p tends to one as $m \to M$.

The right panel of Figure 6 shows the correlation between portfolios as a function of the diversification cost c. Correlation between portfolios steadily increases by reducing diversification costs essentially because the overlap between portfolios increases. It is important to notice that the condition of divergence of the variance does not imply a perfect overlap between portfolio. For example, with the given parameters the transition to infinite variance and non stationary portfolios occurs at $\bar{o} = 0.34$ when $\sigma_s/\sigma_d = 0.3$ and at $\bar{o} = 0.21$ when $\sigma_s/\sigma_d = 0$. Thus correlation between portfolios can become very close to one even if the portfolio overlap is relatively small.

2.5 Bank asset dynamics

The dynamics of the rebalanced bank asset $A^*_{i,t}$, can be written as

$$A^*_{j,t} = \lambda E_{j,t} = \lambda \left(E_{j,t-1} + r^p_{j,t} A^*_{j,t-1} \right) = A^*_{j,t-1} + \lambda r^p_{j,t} A^*_{j,t-1} \tag{32}$$

thus, the relative change of the bank j total asset $r^A_{j,t}$ is simply given as

$$r^A_{j,t} \equiv \frac{A^*_{j,t} - A^*_{j,t-1}}{A^*_{j,t-1}} = \lambda r^p_{j,t}. \tag{33}$$

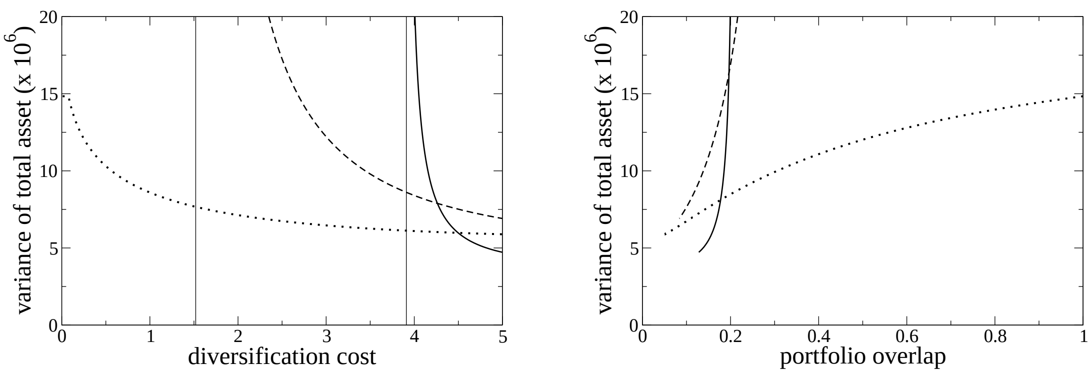


Figure 7: Variance of the total asset, $\sum_{j=1}^{N} r^p_{j,t}$, of the whole banking sector as a function of the diversification cost c (left panel) and of the mean fractional overlap $\bar{o}$ between portfolios (right panel). The used parameters are: $M = 20$, $\alpha = 0.05$, $\mu - r_L = 0.8$, $\gamma = 40$, $\sigma_d = 1$, and $N = 100$. We then choose σ_s equal to 0 (solid line), 0.3 (dashed line), and 0.6 (dotted line). The vertical lines indicate where the variance of total asset diverges.

Therefore, the variance and covariance of the relative change of bank assets $r^A_{j,t}$ are simply

$$V(r^A_{j,t}) = \lambda^2 V(r^p_{j,t}) \tag{34}$$

and

$$Cov(r^A_{h,t}, r^A_{k,t}) = \lambda^2 Cov(r^p_{h,t}, r^p_{k,t}), \tag{35}$$

where the expression for $V(r^p_{j,t})$ and $Cov(r^p_{h,t}, r^p_{k,t})$ are given in equation (28) and (30), respectively. The properties of the bank assets dynamics are then dictated by those of the portfolio (with its exogenous and endogenous components) and further amplified by the degree of leverage.

We can finally compute the variance of the total asset of the whole banking sector

$$V\left(\sum_{j=1}^{N} r^A_{j,t}\right) = NV\left(r^A_{j,t}\right) + N(N-1)Cov\left(r^A_{h,t}, r^A_{k,t}\right) = N\lambda^2 V\left(r^p_{j,t}\right) + N(N-1)\lambda^2 Cov\left(r^p_{h,t}, r^p_{k,t}\right)$$
$$= \lambda^2 V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \tag{36}$$

where $V\left(\sum_{j=1}^{N} r^p_{j,t}\right)$ is explicitly given in terms of the original variables in Appendix D where it is also shown that for $m \to M$ it reduces to

$$V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \xrightarrow[m\to M]{} \frac{N^2\sigma^2_{p_M}}{1-\Lambda_{\max}}. \tag{37}$$

These analytical results allows us to analyze the determinants of the variability of total asset of the banking sector which governs the expansion and contraction of the supply of credit and liquidity to financial system.

Figure 7 shows the variance of the total asset of the whole banking sectors as a function of the diversification cost (left panel) and of the mean fractional overlap between portfolios (right panel). We observe that the variance of the total asset monotonically increases when one decreases diversification cost or increases the overlap between portfolios. As one of these two related variables leads the system close to the critical point, the variance of the total asset of the banking sector explodes. Moreover, close to the transition point, the variance of the total asset increases dramatically when one changes slightly the typical overlap between portfolios.

3 Systemic risk

We now analyze the systemic risk implications of our model first in the static setting without feedback and then in a setting with the endogenous feedback of investor demands on the asset dynamics.

3.1 Static analysis

First, as previously shown, when the diversification m increases, the correlation between the portfolio returns of two financial institutions will increase, with $\rho_p \xrightarrow[m \to M]{} 1$, which, ceteris paribus, tends to increase the probability of a systemwide contagion during a crisis event.

Second, given a negative realization of the systematic (exogenous and endogenous) component $s_t = \bar{e}_t + f_t$, the portfolio return distribution conditioned on this systematic shock s_t^{shock} is (considering, for simplicity, a normal distribution for portfolio returns with zero mean):

$$r_{i,t}^{p}|s_t^{shock} \sim N\left(s_t^{shock},\ \frac{\sigma_d^2}{m}\right). \tag{38}$$

where $r_{i,t}^{p} = \sum_{j=1}^{m} \frac{r_{i,j,t}}{m}$ is the portfolio return of bank i at time t.

Consequently, the probability of default of a financial institution given a systematic shock s_t^{shock} is

$$\begin{aligned} PD_{i,t-1} &= P\left(\left[r_{i,t}^{p}|s_t^{shock}\right] \leq -\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}\right) \\ &= \Phi\left(\frac{-\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} - s_t^{shock}}{\sqrt{\frac{\sigma_d^2}{m}}}\right) \xrightarrow[m \to M,\ M \to \infty]{} 1 \quad \forall\ s_t^{shock} < -\alpha\sigma_s, \end{aligned} \tag{39}$$

where Φ is the standard normal cdf. Therefore, for any negative shock of the systematic component larger than its VaR, i.e. $\alpha\sigma_s$, the probability of default increases with the degree of diversification m.

Notice that a standard measure to evaluate the expected capital shortfall of the firm in a crisis, the Marginal Expected Shortfall (MES) of Acharya et al. (2010) and Brownlees and Engle (2012) does not depend on m. In fact, in our case the MES of institution i would be:

$$MES_{i,t-1}(C) \equiv E_{t-1}(r_{i,t}^{p}|s_t < C) = \frac{Cov(r_{i,t}^{p}, s_t)}{Var(s_t)} E_{t-1}(s_t|s_t < C) = E_{t-1}(s_t|s_t < C)$$

being the portfolio beta equal to one in our simple model. Therefore, although the default probability during a crisis event highly depends on the degree of diversification m, the expected capital shortfall, as measured by the MES, does not account for the effect of diversification, missing this dimension of systemic risk.

Summarizing, higher degree of diversification increases both the probability of default of single institutions (in case of large systematic shocks) and the correlations among them, thus exposing the economy to a higher level of systemic risk.

3.2 Dynamic analysis

The results of the previous section show that the endogenous return dynamics adds an additional component to both the variance and covariance of the risky investments. If such endogenous components were not accounted for in the evaluation of portfolio volatility for the VaR, obviously, there would be an underestimation of each agent's risk, leading to an under capitalization of the banking sector and, hence, to an higher fragility of the system. Nevertheless, the practice of empirically estimating variances and covariances of risky assets from past data, automatically considers both the exogenous and endogenous components of volatility.

However, contrary to the case without endogeneity, the investments variances and covariances now depend on the level of diversification and, in particular, the degree of leverage (through the dynamics of the endogenous component). Therefore, a change, say, in the degree of leverage will cause a structural shift in the future level of variances and covariances which will not be captured by the empirical estimation on past data.

In particular, in periods when leverage increases, portfolio volatility estimated on historical past data will tend to underestimate future risk (coming from stronger rebalancing feedback) leading to an increase of systemic risk. On the contrary, in periods of decreasing leverage, future volatility will tend to decrease (reduced feedback intensity) so that future realized volatility will tend to be lower than the historical one. Therefore, the results of our model provide a theoretical support for countercyclical capital requirements as often advocated in the aftermath of the recent financial crises.

Moreover, it is important to notice that a given negative realization of the exogenous factor f_t, will trigger a sequence of portfolio rebalances causing the price of all risky assets to decay for several periods. Within our framework, we can explicitly compute the expected impact on the future return dynamics triggered by a given common shock.

Being $\boldsymbol{e_t} = \boldsymbol{\Phi r_t}$ (from equation 18), the vector of investment returns also follows a

VAR(1)

$$\boldsymbol{r_t} \;=\; \boldsymbol{e_{t-1}} + \boldsymbol{\iota} f_t + \boldsymbol{\epsilon_t} = \boldsymbol{\Phi} \boldsymbol{r_{t-1}} + \boldsymbol{\iota} f_t + \boldsymbol{\epsilon_t}. \tag{40}$$

The total future impact of the shocks over the next h periods will be given by the h-period cumulative mean return conditioned on the factor shock f_t^{shock}, which is (for h sufficiently large)

$$E\left[\boldsymbol{r_{t:t+h}} | f_t^{shock}\right] \approx \left(\mathbf{I} - \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\iota} f_t^{shock}. \tag{41}$$

Hence, the larger the maximum eigenvalue of $\boldsymbol{\Phi}$ the larger will be the magnitude and persistence of future adjustments leading to a larger cumulative impact that the financial system will have to absorb. So the larger the maximum eigenvalue the higher will be the probability that the system, because of capital or liquidity constraints, will not be able to absorb the initial shock. This also means that systemic risk is positively related to the magnitude of the eigenvalues of the matrix $\boldsymbol{\Phi}$.

4 Discussion: Introducing financial innovations

The results on the dynamics of the asset prices can be summarized as follows. When the costs of diversification c are high, the degree of diversification i.e. the number of asset m randomly selected and the degree of leverage are small. Thus the portfolio of the financial institutions are heterogeneous and little leveraged. Therefore, the endogenous feedbacks, coming from the amplification of individual demands induced by leverage targeting (as illustrated in the previous section), are of moderate size and uncoordinated. Thus, an amplification mechanism at the aggregate level between asset values and prices of risky investments will not tend to arise.

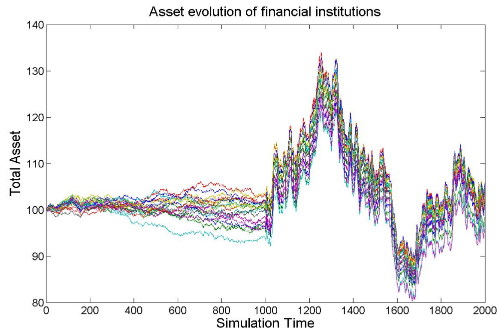
We now discuss the effect of the introduction of financial innovation products (such as securitization of mortgages or ABS products) that permits to reduce the cost of diversification c. Despite the simplicity of our framework, the introduction of financial innovation has several important consequences. First, a financial innovation which reduces the cost of diversification c, by increasing the optimal level of diversification m, reduces the volatility of the portfolio which in turns increase the leverage of the institution. In this way, financial innovation will tend to increase the degree of leverage in the system. By increasing the leverage, the individual exposition to the undiversifiable macro factor risk increases; i.e., although each individual is more resilient to idiosyncratic shocks, they become more sensitive to the shocks in the macro factor.

Second, by increasing m, the overlap in the portfolios of the different financial institutions will be larger, increasing the "similarity" of the portfolio choices among the investors and, thereby, increasing the correlations among portfolio returns and balance sheet dynamics.

Third, an increase in leverage will heavily affect the dynamics of the risky investments by increasing both their variances, covariances and correlations, through a strengthening of the endogenous component.

As a consequence of these effects, individual reactions in terms of asset demands will be more aggressive (due to higher leverage) and more coordinated (because of the larger correlation in the profits-losses realizations). This rise in the strength and coordination of the individual reactions will make more likely to have aggregate feedback in which the rise of the price of some investments leads to an excess of equity (by the realized capital gains) and, hence, to an expansion of the balance sheets driving new demands for the asset which pushes the price up and so on. The very same mechanism will operate also in the opposite direction during market crisis when the aggregate feedback will aggravate price declines and balance sheet contractions. When the diversification cost falls below a given threshold (implying the maximum eigenvalue of the vector return process exceeding one) the aggregate feedback will produce price and balance sheet dynamics that become explosive in a finite time i.e. that have a finite time singularity (as in the somewhat related model of Corsi and Sornette 2010). These feedbacks could be reinforced even further by endogenizing the dynamics of financial innovation or, as in Brunnemeier and Pedersen (2008), that of the market liquidity.

Therefore, through these mechanisms reinforcing the endogenous feedback, financial innovation can give rise to a steep growth (bubble) and plunge (burst) of market prices and banking sector total assets. As explained by Adrian and Shin (2010), the total asset of the banking sector is the relevant variable for the determination of the amount of credit supplied to the financial and real sector. Hence, an increase in the variability of the total asset of the banking sector will have major consequences on the availability of funding to the economy



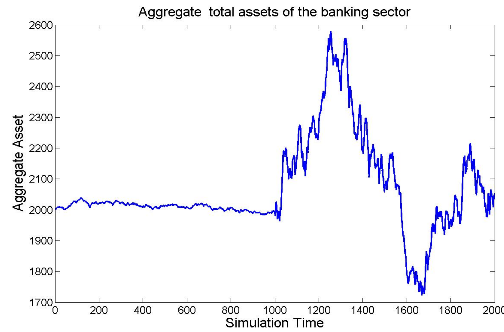


Figure 8: Numerical simulation of the dynamics of individual total asset of financial institutions (left panel) and total asset of the whole banking system (right panel) before and after a structural break (at simulation time 1000) on the diversification costs that induces an increase of leverage and diversification. Leverage goes from 10 to 60 and fractional overlap from 0.1 to 0.8 .

causing the instability to be transmitted from the financial sector to the real one.

To visually illustrate the impact on the dynamics of financial intermediaries total asset of a shift in the degree of leverage and diversification induced by a reduction in the diversification costs, we simulate the bank asset dynamics with a structural break represented by a sudden increase (at simulation time 1000) in the degree of leverage and diversification. Going from a low level of leverage and diversification to a high level we observe: (i) a dramatic increase in the correlation and amplitude of the changes in the total asset of individual financial institutions, see left panel of Figure 8 , and (ii) a sudden shift in the total banking sector assets, which will imply going from an approximately constant supply of credit to a regime with wide swings in the credit supply (right panel of Figure 8).

5 Conclusions

By exploiting basic common practice accounting and risk management rules, we propose a simple analytical dynamical framework to investigate the effects of micro-prudential changes on macro-prudential outcomes. Specifically, we study the consequence of the introduction of a financial innovation that allows reducing the cost of portfolio diversification in a financial system populated by financial institutions having capital requirements in the form of VaR constraint and following standard mark-to-market and risk management rules. We provide a fully analytical quantification of the multivariate feedback effects between investment prices and bank behavior induced by portfolio rebalancing and show how changes in the constraints of the bank portfolio optimization endogenously drive the dynamics of the balance sheet aggregate of financial institutions and, thereby, the availability of bank liquidity to the economic system and systemic risk.

The analytical results obtained by applying our simple framework are manifolds: (i) a reduction of diversification costs, by increasing the level of diversification and hence relaxing the VaR constraint, allows the financial institutions to increase the optimal leverage; (ii) it also increases the degree of overlap, and thereby correlation, between the portfolios of financial institutions; (iii) even in absence of feedback effects, higher degree of diversification increases both the probability of default of single institutions (in case of large systematic shocks) and the correlations among them, thus exposing the economy to a higher level of systemic risk; (iv) the higher overlap induced by a reduction in diversification costs increases both the variance and correlation of the investment demands of financial institutions rebalancing their portfolios; (v) the dynamic interaction between investment prices and bank behavior induced by portfolio rebalancing leads to a multivariate VAR process whose maximum eigenvalue depends on the degree of leverage and on the average illiquidity of the assets; (vi) higher diversification, by increasing the strength and coordination of individual feedbacks, can lead to dynamic instability of the system; (vii) the VAR process can be represented as a combination of many idiosyncratic AR processes around a single common AR process of the average values; (viii) the endogenous feedback introduces an additional component to the variance, covariance and correlation of both the individual investment assets and the bank portfolios; (ix) both the variance and correlation of individual investments monotonically increase with a reduction in the diversification costs; (x) a simple variance multiplier exists for the variance of the observed market portfolio. (xi) the relation between the variance of the portfolio and diversification costs is non-monotonic as it initially declines with costs while then rapidly increases when the reduction of diversification costs makes the system approaching its critical point causing the variance to explode; (xii) the effects of the endogenous feedback make historical estimation of variances and covariances to be overestimated during periods of increasing leverage and underestimated during periods of deleveraging, thus providing a rationale for the adoption of countercyclical capital requirements; (xiii) in presence of endogenous feedbacks, a negative realization of the systematic component will trigger a sequence of portfolio rebalances which will amplify, over time, its initial impact; (xiv) the variability of total asset of the banking sector, which governs the expansion and contraction of the supply of credit and liquidity to financial system, is highly sensitive to variation in the costs of diversification.

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Appendices

A Portfolio rebalance

Given that $A^*_{j,t} = \lambda E_{j,t}$, the difference between the desired amount of asset ($A^*_{j,t}$) and the actual one ($A_{j,t}$) can be written as (dropping the sub-index j for sake of notation simplicity):

$$\begin{aligned}
\Delta R_t &\equiv A^*_t - A_t \\
&= \lambda E_t - A_t \\
&= \lambda(E_{t-1} + r^p_t A^*_{t-1}) - (1 + r^p_t) A^*_{t-1} \\
&= \lambda \left(\frac{A^*_{t-1}}{\lambda} + r^p_t A^*_{t-1} \right) - (1 + r^p_t) A^*_{t-1} \\
&= (\lambda - 1) r^p_t A^*_{t-1}
\end{aligned}$$

B Derivation for the demand function

B.1 Approximation of the demand function

We start from the expression of Eq. (12) for the demand $D_{i,t}$,

$$D_{i,t} = \sum_{j=1}^{N} I_{\{i \in j\}} (\lambda - 1) r^p_{j,t} \frac{A^*_{j,t-1}}{m} \tag{42}$$

where $I_{\{i \in j\}}$ is the indicator function which takes value one when investment i is in the portfolio of institution i and zero otherwise.

We assume that total assets are approximately the same for all the banks, i.e. $A^*_{j,t} \simeq A^*_t$, and thus

$$D_{i,t} \approx (\lambda - 1) \frac{A^*_{t-1}}{m} \sum_{j=1}^{N} I_{\{i \in j\}} r^p_{j,t}. \tag{43}$$

Since

$$r^p_{j,t} = \frac{1}{m} \sum_{k=1}^{M} I_{k \in j} r_{k,t} \tag{44}$$

the sum in the above expression for D can be rewritten as

$$\frac{1}{m} \sum_{j=1}^{N} I_{\{i \in j\}} \sum_{k=1}^{M} I_{k \in j} r_{k,t} \tag{45}$$

In order to calculate this term, we need to consider all the banks having investment i in their portfolio and then, for each investment k (including i) we have to count the number of them having investment k in their portfolio. This is of course a random variable and we use averages. On average there are Nm/M banks having investment i in the portfolio. All of them have (by definition) investment i in the portfolio. On the other hand, for an investment $k \neq i$, we can consider the restricted bipartite network of Nm/M banks, each having $m-1$ investments among a universe of $M-1$ (we know with certainty that they have investment i). Therefore the number of banks having investment i in their portfolio and having also investment k is $\frac{Nm}{M}\frac{m-1}{M-1}$. Thus the sum in Eq. (45) can be rewritten as

$$\frac{N}{M} \left(r_{i,t} + \frac{m-1}{M-1} \sum_{k \neq i} r_{k,t} \right) \tag{46}$$

and the demand is

$$D_{i,t} \approx (\lambda - 1) \frac{A^*_{t-1}}{m} \frac{N}{M} \left(r_{i,t} + \frac{m-1}{M-1} \sum_{k \neq i} r_{k,t} \right) \tag{47}$$

i.e. Eq. (13).

B.2 Demand correlation

The variance of the demand of investment i, D_i is,

$$V(D_i) = \frac{1}{m^2}\frac{N^2}{M^2}\left[\left(1+(M-1)\left(\frac{m-1}{M-1}\right)^2\right)(\sigma_f^2+\sigma_\epsilon^2)+ \right.$$
$$\left. + \left(2(M-1)\frac{m-1}{M-1}+(M^2-3M+2)\left(\frac{m-1}{M-1}\right)^2\right)\sigma_f^2\right] \tag{48}$$

and the covariance between demand of investment i and j is,

$$Cov(D_i, D_j) = \frac{1}{m^2}\frac{N^2}{M^2}\left[\left(2\frac{m-1}{M-1}+(M-2)\left(\frac{m-1}{M-1}\right)^2\right)(\sigma_f^2+\sigma_\epsilon^2)+ \right.$$
$$\left. + \left(1+2(M-2)\frac{m-1}{M-1}+(M^2-3M+3)\left(\frac{m-1}{M-1}\right)^2\right)\sigma_f^2\right]. \tag{49}$$

Hence, the correlation becomes

$$\rho(D_i, D_j) = \frac{\left(2\frac{m-1}{M-1}+(M-2)\left(\frac{m-1}{M-1}\right)^2\right)\left(\sigma_f^2+\sigma_\epsilon^2\right)+\left(1+2(M-2)\frac{m-1}{M-1}+(M^2-3M+3)\left(\frac{m-1}{M-1}\right)^2\right)\sigma_f^2}{\left(1+(M-1)\left(\frac{m-1}{M-1}\right)^2\right)\left(\sigma_f^2+\sigma_\epsilon^2\right)+\left(2(M-1)\frac{m-1}{M-1}+(M^2-3M+2)\left(\frac{m-1}{M-1}\right)^2\right)\sigma_f^2} \xrightarrow[m\to M]{} 1 \tag{50}$$

C VAR Eigenvalues

In this Appendix we derive the eigenvalues of $\mathbf{\Psi}$ and (in an approximate form) of $(\lambda-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi}$. First of all we notice that the matrix $\mathbf{\Psi}$ has all diagonal elements equal to $d = 1/m$ and all the off diagonal elements equal to $d_{\text{off}} = \frac{1}{m}\frac{m-1}{M-1}$. Thus, $(\lambda-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi} - \mathbf{I}\Lambda$ can be rewritten as[6]

$$(\lambda-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi} - \mathbf{I}\Lambda = A + uv' \tag{51}$$

where

$$A = diag[g_i(d-d_{\text{off}}) - \Lambda] \tag{52}$$
$$u = (g_1 \dots g_M)' \tag{53}$$
$$v = (d_{\text{off}} \dots d_{\text{off}})' \tag{54}$$

[6]Please note that we denote the eigenvalue with Λ while the common notation λ is devoted, throughout the paper, to denote the leverage.

where we have set $g_i = (\lambda - 1)\gamma_i^{-1}$. In order to compute the characteristic polynomial of the matrix we can use the Sherman-Morrison formula

$$\det(A+uv') = (1+v'A^{-1}u)\det A = \left(1 + d_{\text{off}} \sum_{i=1}^{M} \frac{g_i}{g_i(d - d_{\text{off}}) - \Lambda}\right) \prod_{i=1}^{M} [g_i(d-d_{\text{off}})-\Lambda] \quad (55)$$

Setting this expression to zero and solving for Λ gives the eigenvalues, but the equation cannot be solved analytically in general.

If all the liquidity parameters γ_i are equal to γ, the above expression simplifies to

$$[g(d - d_{\text{off}}) - \Lambda]^{M-1}[g(d - d_{\text{off}}) - \Lambda + bMg] = 0 \quad (56)$$

Thus in the degenerate case, the spectrum is composed by $M - 1$ degenerate (and small) eigenvalues equal to $\frac{g}{m}\frac{M-m}{M-1}$ and one large eigenvalue equal to $g = (\lambda - 1)\gamma^{-1}$.

When the liquidity parameters are different, we can expect that the spectrum has the same characteristics and the large eigenvalue is determined by setting to zero the first term in brackets of Equation 55, i.e.

$$1 + d_{\text{off}} \sum_{i=1}^{M} \frac{g_i}{g_i(d - d_{\text{off}}) - \Lambda} = 0 \quad (57)$$

Since the eigenvalue is large, we can approximate this equation with $1 - d_{\text{off}} \sum_i g_i/\Lambda \simeq 0$, i.e.

$$\Lambda \simeq d_{\text{off}} \sum_{i=1}^{M} g_i = (\lambda - 1)\frac{m-1}{m}\frac{M}{M-1}\overline{\gamma^{-1}} \simeq (\lambda - 1)\overline{\gamma^{-1}} \quad (58)$$

where

$$\overline{\gamma^{-1}} = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{\gamma_i} \quad (59)$$

is the average of the inverse of the liquidity parameters. For a discussion of the validity of this approximation, see Lillo and Mantegna (2005)

D Endogenous variance and covariance

Recalling that

$$\begin{aligned} e_{i,t} &= (1-b)\,a(e_{i,t-1} + \varepsilon_{i,t}) + b\,Ma(\bar{e}_{t-1} + \bar{\varepsilon}_t), \\ \bar{e}_t &= a(1 - b + bM)(\bar{e}_{t-1} + \bar{\varepsilon}_t) \equiv \phi(\bar{e}_{t-1} + \bar{\varepsilon}_t), \end{aligned}$$

with scalar $a = \frac{\lambda-1}{m\gamma}$, $b = \frac{m-1}{M-1}$, and $\phi = a(1 - b + bM)$, and that stationarity of e_t implies $\gamma > \lambda - 1$, we have that

$$V(\bar{e}_t) = \frac{\phi^2 V(\bar{\varepsilon})}{1-\phi^2} = \frac{(\lambda-1)^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{\gamma^2 - (\lambda-1)^2} = \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2}\left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}\right) \quad (60)$$

and

$$\begin{aligned} Cov(e_{i,t}, \bar{e}_t) &= \phi abMV(\bar{e}_t) + \phi a(1-b)Cov(e_{i,t}, \bar{e}_t) + \phi^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}) \\ &= \frac{\phi abMV(\bar{e}_t) + \phi^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{1 - \phi a(1-b)} \\ &= \frac{(\lambda-1)^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{\gamma^2 - (\lambda-1)^2} \\ &= \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2} \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}\right) \\ &= V(\bar{e}_t). \end{aligned} \tag{61}$$

Hence, the variance of $e_{i,t}$ reads

$$\begin{aligned} V(e_{i,t}) &= a^2(1-b)^2V(e_{i,t}) + a^2b^2M^2V(\bar{e}_t) + 2a^2b(1-b)MCov(e_{i,t}, \bar{e}_t) + \\ &+ a^2(1-b)^2(\sigma_f^2 + \sigma_\epsilon^2) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \sigma_\epsilon^2/M) \\ &= \frac{\left(a^2b^2M^2 + 2a^2b(1-b)M\right)V(\bar{e}_t) + a^2(1-b)^2(\sigma_f^2+\sigma_\epsilon^2) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{1-a^2(1-b)^2} \end{aligned} \tag{62}$$

and the covariance between two different stocks is

$$\begin{aligned} Cov(e_{i,t}, e_{j,t}) &= a^2(1-b)^2Cov(e_{i,t}, e_{j,t}) + a^2b^2M^2V(\bar{e}_t) + 2a^2b(1-b)MCov(e_{i,t}, \bar{e}_t) \\ &+ a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M} + a^2(1-b)^2\sigma_f^2) \\ &= \frac{\left(a^2b^2M^2 + 2a^2b(1-b)M\right)V(\bar{e}_t) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}) + a^2(1-b)^2\sigma_f^2}{1-a^2(1-b)^2} \end{aligned} \tag{63}$$

By substituting back a, b, ϕ, and $V(\bar{e}_t)$, and defining $\tilde{\lambda} = \lambda - 1$ we get the expression of the variance and covariance of $e_{i,t}$ in terms of the original variables:

$$V(e_{i,t}) = -\frac{\tilde{\lambda}^2\Big(m^2\Big(\sigma_\epsilon^2\left(\tilde{\lambda}^2-\gamma^2(M-1)\right)+\sigma_f^2\left(\tilde{\lambda}^2-\gamma^2(M-1)^2\right)\Big)+2m\Big(M\left(\sigma_\epsilon^2\left(\gamma^2-\tilde{\lambda}^2\right)-\tilde{\lambda}^2\sigma_f^2\right)-\gamma^2\sigma_\epsilon^2\Big)+M\Big(M\left(\sigma_\epsilon^2\left(\tilde{\lambda}^2-\gamma^2\right)+\tilde{\lambda}^2\sigma_f^2\right)+\gamma^2\sigma_\epsilon^2\Big)\Big)}{\left(\gamma^2-\tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)+2\tilde{\lambda}^2mM-\tilde{\lambda}^2M^2\right)} \tag{64}$$

$$Cov(e_{i,t}, e_{j,t}) = -\frac{\tilde{\lambda}^2\left(m^2\left(\sigma_f^2\left(\tilde{\lambda}^2-\gamma^2(M-1)^2\right)-\gamma^2(M-2)\sigma_\epsilon^2\right)-2m\left(\tilde{\lambda}^2M\sigma_f^2+\gamma^2\sigma_\epsilon^2\right)+M\left(\tilde{\lambda}^2M\sigma_f^2+\gamma^2\sigma_\epsilon^2\right)\right)}{\left(\gamma^2-\tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)+2\tilde{\lambda}^2mM-\tilde{\lambda}^2M^2\right)} \tag{65}$$

and that of the correlations

$$Corr(e_{i,t}, e_{j,t}) = \frac{m^2\left(\gamma^2(M-2)\sigma_\epsilon^2+\sigma_f^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)\right)+2m\left(\tilde{\lambda}^2M\sigma_f^2+\gamma^2\sigma_\epsilon^2\right)-M\left(\tilde{\lambda}^2M\sigma_f^2+\gamma^2\sigma_\epsilon^2\right)}{m^2\left(\sigma_\epsilon^2\left(\gamma^2(M-1)-\tilde{\lambda}^2\right)+\sigma_f^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)\right)+2m\left(M\left(\sigma_\epsilon^2\left(\tilde{\lambda}^2-\gamma^2\right)+\tilde{\lambda}^2\sigma_f^2\right)+\gamma^2\sigma_\epsilon^2\right)+M\Big(M\left(\sigma_\epsilon^2(\gamma^2-(\lambda-1)^2)-\tilde{\lambda}^2\sigma_f^2\right)-\gamma^2\sigma_\epsilon^2\Big)} \tag{66}$$

for which we can prove $Corr(e_{i,t}, e_{j,t}) \xrightarrow[m \to M]{} 1$.

Finally, we compute the variance of the sum of all the portfolios held by the N banks. Using equation (28) and (30) for $V\left(r^p_{j,t}\right)$ and $Cov\left(r^p_{h,t}, r^p_{k,t}\right)$, we have

$$\begin{aligned} V\left(\sum_{j=1}^{N} r^p_{j,t}\right) =& NV\left(r^p_{j,t}\right) + N(N-1)Cov\left(r^p_{h,t}, r^p_{k,t}\right) \\ &= N(m^3\left(\tilde{\lambda}^2\sigma_\epsilon\left(-\gamma^2+\tilde{\lambda}^2+N\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)\right)+\gamma^2 MN\sigma_f\left(\gamma^2(M-1)^2-(\lambda-1)^2\right)\right)+m^2M \\ &\left(\sigma_\epsilon\left(3\tilde{\lambda}^2\left(\gamma^2-(\lambda-1)^2\right)+N\left(3\tilde{\lambda}^4-\gamma^2\tilde{\lambda}^2\left(M^2-2M+2\right)+\gamma^4(M-1)^2\right)\right)+2\gamma^2\tilde{\lambda}^2MN\sigma_f\right) \\ &-\tilde{\lambda}^2 mM^2\left(\gamma^2MN\sigma_f+\sigma_\epsilon\left(3\left(\gamma^2-\tilde{\lambda}^2\right)+N\left(3\tilde{\lambda}^2-2\gamma^2\right)\right)\right)+\tilde{\lambda}^2M^3(N-1)\sigma_\epsilon\left(\tilde{\lambda}^2-\gamma^2\right))/ \\ &mM\left(\gamma^2-\tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)+2\tilde{\lambda}^2mM-\tilde{\lambda}^2M^2\right) \end{aligned} \tag{67}$$

which reduces to

$$V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \xrightarrow[m\to M]{} \frac{\gamma^2N^2(\sigma_f+\frac{\sigma_\epsilon}{M})}{\left(\gamma^2-\tilde{\lambda}^2\right)} = \frac{N^2\sigma^2_{p_M}}{1-\Lambda_{\max}} \tag{68}$$