Portugal. Math. (N.S.)
Vol. 67, Fasc. 3, 2010, 321–367
DOI 10.4162/PM/1868

Portugaliae Mathematica

Interest rate theory and geometry

Tomas Björk* and Raquel M. Gaspar**

(Communicated by Luis Nunes Vicente)

Abstract. In this paper we provide an overview of some basic topics in interest rate theory from the point of view of arbitrage free pricing. We cover short rate models, affine term structure models, inversion of the yield curve and the Musiela parameterization. We treat geometric interest rate theory in some detail, and we also review the potential approach to positive interest rates. The text is essentially self-contained, and references to the literature can be found in Section 6.

Mathematics Subject Classification (2010). Primary 60-02, 51-02, 58-02, 35-02, 34-02; Secondary 60G07, 51E25, 58J65, 35G30, 34B15, 34B60.

Keywords. Interest rate models, arbitrage theory, stochastic processes, martingales, PDEs, ODEs, manifolds, potentials, finite dimensional realizations.

1. General background

In this paper we give an overview of some basic topics in interest rate theory from the point of view of arbitrage free pricing. Not all proofs are given, but the interested reader can find the relevant references to the literature in Section 6.

We consider a financial market model on a finite time interval $[0, \hat{T}]$ living on a filtered probability space $(\Omega, \mathscr{F}, \boldsymbol{F}, P)$ where $\boldsymbol{F} = \{\mathscr{F}_t\}_{t \geq 0}$ and P is interpreted as the "objective" or "physical" probability measure. The basis is assumed to carry a standard m-dimensional Wiener process W, and we also assume that the filtration $\boldsymbol{F}$ is the internal one generated by W. The choice of a Wiener filtration is made for convenience, and the theory below can be extended to a general semimartingale framework.

*Support from the Tom Hedenius and Jan Wallander Foundation is gratefully acknowledged. Both authors are much indebted to an anonymous referee for a number of very helpful comments.

**Financial support of XXX and of the Portuguese Science Foundation—FCT—under grant PTDC/MAT/64838/2006 is gratefully acknowledged.

We assume that there exist $N+1$ non-dividend-paying assets on the market, and the prices at time t of these assets are denoted by $S_0(t), S_1(t), \dots, S_N(t)$. We assume that the price processes are Itô processes and that $S_0(t) > 0$ with probability one. We view the price vector process $S = (S_0, S_1, \dots S_N)^*$ as a column vector process, where * denotes transpose.

A *portfolio* is any adapted (row vector) process $h = (h^0, h^1, \dots, h^N)$, where we interpret h_t^i as the number of units that we hold of asset i in the portfolio at time t. The corresponding market *value* process V^h is defined by $V^h(t) = h(t)S(t) = \sum_{i=0}^N h^i(t)S_i(t)$, and the portfolio is said to be *self-financing* if the condition $dV(t) = h(t)\,dS(t)$ is satisfied.

An *arbitrage* possibility is a self-financing portfolio h with the properties that $V^h(0) = 0$, $P\big(V^h(T) \geq 0\big) = 1$ and $P\big(V^h(T) > 0\big) > 0$. An arbitrage would constitute a "money-making machine" and a minimal requirement of market efficiency is that the market is free of arbitrage possibilities. The main result in this direction is, subject to some technical conditions, as follows.

Theorem 1.1. *The market is free of arbitrage if and only if there exists a probability measure Q with the properties*

(1) *$Q \sim P$,*

(2) *all* normalized *asset processes*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \dots, \frac{S_N(t)}{S_0(t)}$$

are Q-martingales.

Such a measure Q (which is typically not unique, see below) is called a *martingale measure*. The *numeraire asset* S_0 could in principle be any asset with positive prices, but very often it is chosen as the *money account* B defined by $dB(t) = r(t)B(t)\,dt$ where r is the short interest rate, i.e.,

$$B(t) = e^{\int_0^t r(s)\,ds}.$$

A *contingent T-claim* is any random variable $Y \in \mathscr{F}_T$, where the interpretation is that the holder of the claim will receive the stochastic amount Y (in a given currency) at time T. Given a T-claim Y, a self-financing portfolio h is said to *replicate* (or "hedge against") Y if $V^h(T) = Y$, P-a.s. The market model is *complete* if every claim can be replicated. The main result for completeness in an arbitrage free market is the following.

Theorem 1.2. *The market is complete if and only if the martingale measure is unique.*

We now turn to the pricing problem for contingent claims. In order to do this, we consider the "primary" market $S_0, S_1, \ldots, S_N$ as given *a priori*, and we fix a T-claim Y. Our task is that of determining a "reasonable" price process $\Pi(t; Y)$ for Y, and we assume that the primary market is arbitrage free. There are two main approaches:

- The derivative should be priced in a way that is *consistent* with the prices of the underlying assets. More precisely we should demand that the extended market $\Pi(t; Y), S_0(t), S_1(t), \ldots, S_N(t)$ is free of arbitrage possibilities.
- If the claim is *attainable*, with hedging portfolio h, then the only reasonable price is given by $\Pi(t; Y) = V(t; h)$.

In the first approach above, we thus demand that there should exist a martingale measure Q for the extended market $\Pi(t; Y), S_0(t), S_1(t), \ldots, S_N(t)$. Letting Q denote such a measure, assuming enough integrability, and applying the definition of a martingale measure we obtain

$$\frac{\Pi(t; Y)}{S_0(t)} = E^Q\left[\frac{\Pi(T; Y)}{S_0(T)}\,\middle|\,\mathcal{F}_t\right] = E^Q\left[\frac{Y}{S_0(T)}\,\middle|\,\mathcal{F}_t\right].$$

We thus have the following result.

Theorem 1.3 (General Pricing Formula). *The arbitrage free price process for the T-claim Y is given by*

$$\Pi(t; Y) = S_0(t) E^Q\left[\frac{Y}{S_0(T)}\,\middle|\,\mathcal{F}_t\right], \tag{1}$$

where Q is a (*not necessarily unique*) *martingale measure for the a priori given market $S_0, S_1, \ldots, S_N$, with S_0 as the numeraire.*

Note that different choices of Q will generically give rise to different price processes.

In particular we note that if we assume that if S_0 is the money account

$$S_0(t) = S_0(0) \cdot e^{\int_0^t r(s)\,ds},$$

where r is the short rate, then (1) reduces to the familiar "risk neutral valuation formula".

Theorem 1.4 (Risk Neutral Valuation Formula). *Assuming the existence of a short rate, the pricing formula takes the form*

$$\Pi(t; Y) = E^Q[e^{-\int_t^T r(s)\,ds} Y \mid \mathcal{F}_t].$$

where Q is a (not necessarily unique) martingale measure with the money account as the numeraire.

For the second approach to pricing let us assume that Y can be replicated by h. Since the holding of the derivative contract and the holding of the replicating portfolio are equivalent from a financial point of view, we see that the price of the derivative must be given by the formula

$$\Pi(t; Y) = V^h(t). \tag{2}$$

One problem here is what will happen in a case when Y can be replicated by two different portfolios, and one would also like to know how this formula is connected to (1).

Defining $\Pi(t; Y)$ by (2) we note that the process $\Pi(t; Y)/S_0(t)$ is a normalized asset price and thus a Q-martingale. Consequently we again obtain the formula (1) and for an attainable claim we have in particular the formula

$$V^h(t) = S_0(t)E^Q\left[\frac{Y}{S_0(T)}\middle|\mathscr{F}_t\right],$$

which will hold for any replicating portfolio and for any martingale measure Q. Thus we see that the two pricing approaches above do in fact coincide on the set of attainable claims.

We finish with a remark on the characterization of a risk neutral martingale measure.

Lemma 1.5. *A risk neutral martingale measure, i.e., an EMM with the bank account as numeraire, is characterized by the properties that $Q \sim P$, and that every asset price process has the short rate as its local rate of return under Q. More precisely, under Q the dynamics of any asset price process π (derivative or underlying) must be of the form*

$$d\pi_t = \pi_t r_t\,dt + \pi_t \sigma_t^\pi\,dW_t^Q, \tag{3}$$

where r is the short rate and W^Q is Q-Wiener.

2. Interest rates and the bond market

Our main object of study is the zero coupon bond market, and we need some formal definitions.

Definition 2.1. A *zero coupon bond* with *maturity date* T, also called a T-bond, is a contract which guarantees the holder \$1.00 to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$.

Given the bond market above, one can define a (surprisingly large) number of *riskless interest rates*. The term LIBOR below, is an acronym for "London Interbank Offered Rate".

Definition 2.2. (1) The *continuously compounded forward rate for* $[\boldsymbol{S},\boldsymbol{T}]$ *contracted at* $\boldsymbol{t}$ is defined as

$$R(t;S,T) = -\frac{\log p(t,T) - \log p(t,S)}{T-S}.$$

(2) The *continuously compounded spot rate for* $[\boldsymbol{S},\boldsymbol{T}]$ is defined as

$$R(S,T) = -\frac{\log p(S,T)}{T-S}.$$

(3) The *instantaneous forward rate with maturity* $\boldsymbol{T}$*, contracted at* $\boldsymbol{t}$*,* is defined by

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}.$$

(4) The *instantaneous short rate at time* $\boldsymbol{t}$ is defined by

$$r(t) = f(t,t).$$

We now go on to define the money account process B.

Definition 2.3. The *money account* process is defined by

$$B(t) = e^{\int_0^t r(s)\,ds},$$

i.e.,

$$dB(t) = r(t)B(t)\,dt, \qquad B(0) = 1.$$

The interpretation of the money account is that you may think of it as describing a bank with the stochastic short rate r.

As an immediate consequence of the definitions we have the following useful formulas.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 325)

Lemma 2.4. *For $t \le s \le T$ we have*

$$p(t,T) = p(t,s) \cdot e^{-\int_s^T f(t,u)\,du},$$

and in particular

$$p(t,T) = e^{-\int_t^T f(t,u)\,du}.$$

We finish this section by presenting the relations that hold between the dynamics of forward rates and those of the corresponding bond prices. These relations will be used repeatedly below. We will consider dynamics of the following form.

Bond price dynamics:

$$dp(t,T) = p(t,T)m(t,T)\,dt + p(t,T)v(t,T)\,dW(t). \tag{4}$$

Forward rate dynamics:

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t). \tag{5}$$

The Wiener process W is allowed to be vector valued, in which case the volatilities $v(t,T)$ and $\sigma(t,T)$ are row vectors. The processes $m(t,T)$, $v(t,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are allowed to be arbitrary adapted processes parameterized by time of maturity T.

Our main technical tool is as follows.

Proposition 2.5. *If $f(t,T)$ satisfies* (5) *then $p(t,T)$ satisfies*

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt + p(t,T)S(t,T)\,dW(t),$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)\,ds, \\ S(t,T) = -\int_t^T \sigma(t,s)\,ds. \end{cases}$$

3. Factor models

Since the price of a contingent claim Y is given by the general formula

$$\Pi(t;Y) = E^Q[e^{-\int_t^T r_s\,ds}\,Y \mid \mathscr{F}_t],$$

it is natural to study Markovian factor models of the form

$$\begin{aligned} dX_t &= \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ r_t &= h(t, X_t), \end{aligned}$$

where μ, σ, and h are given deterministic functions and W is Wiener. In this framework we typically restrict ourselves to contingent T-claims Y of the form $Y = \Phi(X_T)$, where Φ denotes the *contract function*, i.e., Φ specifies the amount of money to be paid to the holder of the contract at time T. This modeling can be done either under the objective measure P, or under a martingale measure Q.

We recall that the defining properties of a risk neutral martingale measure Q are that $Q \sim P$ and that Π_t/B_t should be a Q-martingale for every asset price process Π. Since, in the present setup, the only asset price specified a priori is the bank account B_t, and since $B_t/B_t = 1$ is trivially a Q-martingale, we see that in this case *every* measure $Q \sim P$ is a martingale measure, and that a particular choice of Q will generate arbitrage free asset prices by the prescription

$$\Pi(t; Y) = E^Q[e^{-\int_t^T h(s, X_s)\,ds}\Phi(X_T) \mid \mathscr{F}_t], \tag{6}$$

for any claim Y of the form $Y = \Phi(X_T)$.

3.1. Modeling under the objective measure *P*. As above we consider a factor model of the form

$$\begin{aligned} dX_t &= \mu^p(t, X_t)\,dt + \sigma(t, X_t)\,dW_t^p, \\ r_t &= h(t, X_t), \end{aligned}$$

where W^p is P-Wiener. The price of claim Y of the form $Y = \Phi(X_T)$ is again given by the formula (6) above. In the present setting, with the filtration generated by W^p, it follows that the likelihood process L defined by

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathscr{F}_t$$

is obtained by a Girsanov transformation of the form

$$dL_t = L_t \varphi_t^\star\,dW_t^p, \qquad L_0 = 1.$$

Here and in the following $\star$ denotes transpose. To keep the Markovian structure we now assume that the Girsanov kernel process φ is of the form $\varphi(t, X_t)$ and from the Girsanov Theorem we can write $dW_t^p = \varphi_t\,dt + dW_t$ where W is Q-Wiener. We thus have the Q-dynamics of X as

$$dX_t = \{\mu^p(t, X_t) + \sigma(t, X_t)\varphi(t, X_t)\}\,dt + \sigma(t, X_t)\,dW_t.$$

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 327)

For notational simplicity we denote the Q-drift of X by μ, i.e.,

$$\mu(t,x) = \mu^p(t,x) + \sigma(t,x)\varphi(t,x).$$

Since the price process $\Pi(t;Y)$ for a claim of the form $Y = \Phi(X_T)$ is given by (6) we now have the following result, which follows directly from the Kolmogorov backward equation.

Theorem 3.1. • *For a claim of the form $Y = \Phi(X_T)$, the price process $\Pi(t;Y)$ is in fact of the form $\Pi(t;Y) = F(t,X_t)$ where F satisfies the* term structure equation

$$\frac{\partial F}{\partial t}(t,x) + \mathscr{A}F(t,x) - h(t,x)F(t,x) = 0, \tag{7}$$

$$F(T,x) = \Phi(x), \tag{8}$$

where the operator $\mathscr{A}$ is given by

$$\mathscr{A}F(t,x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial F}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x)$$

and where $C(t,x) = \sigma(t,x)\sigma(t,x)^{\star}$.

• *In particular, bond prices are given by $p(t,T) = F^T(t,X_t)$ (the index T is viewed as a parameter), where the pricing function F^T satisfies*

$$\frac{\partial F^T}{\partial t}(t,x) + \mathscr{A}F^T(t,x) - h(t,x)F^T(t,x) = 0, \tag{9}$$

$$F^T(T,x) = 1. \tag{10}$$

3.2. The market price of risk. There is an immediate economic interpretation of the Girsanov kernel φ above. To see this let π_t be the price process of any asset (derivative or underlying) in the model. We write the P-dynamics of π as

$$d\pi_t = \pi_t\alpha_t\,dt + \pi_t\delta_t\,dW_t^P,$$

where α is the local mean rate of return of π (under P) and δ is the (vector) volatility process. From the Girsanov Theorem we obtain, as above,

$$d\pi_t = \pi(t)\{\alpha_t + \delta_t\varphi_t\}\,dt + \pi_t\delta_t\,dW_t,$$

where W is Q-Wiener. From Lemma 1.5 we have , on the other hand,

$$d\pi_t = \pi_t r_t\,dt + \pi_t\delta_t\,dW_t,$$

so we obtain the relation

$$\alpha_t + \delta_t \varphi_t = r_t,$$

or, equivalently,

$$\alpha_t - r_t = -\delta_t \varphi_t = -\sum_i \delta_{it} \varphi_{it}.$$

In other words, the *risk premium* for π, given by $\alpha_t - r_t$, i.e., the excess rate of return above the risk free rate r, is given (apart from a minus sign) as the sum of the volatility terms δ_i multiplied by the "factor loadings" φ_i. This has motivated economists to refer to the process $\lambda_t = -\varphi_t$ as the "market price of risk" process, where λ_i is the market price of risk for Wiener factor number i. In particular we see that if W (and thus δ) is scalar then λ in fact equals the *Sharpe ratio*, i.e.,

$$\lambda_t = \frac{\alpha_t - r_t}{\delta_t}.$$

The economic interpretation is that λ is a measure of the aggregate risk aversion in the market, in the sense that if λ is positive then the market is risk averse, if λ is negative then the market is risk loving and if $\lambda = 0$ is positive then the market is risk neutral. We summarize the moral in the following slogan.

Result 3.2. The martingale measure is chosen by *the market*.

3.3. Martingale modeling. In order to construct a factor model of the type above, and to be able to compute derivative prices, it seems that we have to model the following objects.

- The P-drift μ^p.
- The volatility σ (which is the same under P and under Q.
- The market price of risk $\lambda = -\varphi$, which connects Q to P by a Girsanov transformation.

However, from the pricing formula (6) we have the following simple observation.

Proposition 3.3. *The term structure of bond prices, as well as the prices of all other derivatives, are completely determined by specifying the dynamics of X under the martingale measure Q.*

This observation has led to the following standard modeling procedure: instead of specifying μ^p, σ and λ under the objective probability measure P we will hence-

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 329)

forth specify the dynamics of the factor process X directly under the martingale measure Q. This procedure is known as *martingale modeling*, and the typical assumption will thus be that X under Q has dynamics given by

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \tag{11}$$

where W is Q-Wiener. The short rate is as before defined by

$$r_t = h(t, X_t). \tag{12}$$

The pricing formulas from Theorem 3.1 still hold.

3.4. Affine term structures. In order to compute bond prices, we have to be able so solve the term structure equation (9)–(10). It turns out that the only cases when the term structure equation can be solved analytically is more or less when we have an *affine term structure.*

Definition 3.4. The factor model (11)–(12) above is said to possess an affine term structure (ATS for short) if bond prices are of the form

$$p(t, T) = e^{A(t,T) - B(t,T)X_t}, \tag{13}$$

where A (scalar) and B (row vector) are deterministic functions of t and T.

The importance of the ATS models stem from the fact that these are roughly speaking the only models for which we can obtain analytical formulas for bond prices and bond option prices. The question now arises as to when we have an ATS, and the basic result is as follows.

Theorem 3.5. *Sufficient conditions for the existence of an affine term structure are the following.*

(1) *The drift (under Q) is an affine function of the factors, i.e., μ is of the form*

$$\mu(t, x) = \alpha(t) + \Delta(t)x,$$

where the $n \times 1$ column vector α and the $n \times n$ matrix Δ are deterministic functions of time.

(2) *The "square of the diffusion" is an affine function of the factors, i.e., $\sigma\sigma^\star$ is of the form*

$$\sigma(t, x)\sigma(t, x)^\star = C(t) + \sum_{i=1}^{n} D_i(t)x_i,$$

where C and D_i are deterministic $n \times n$ matrix functions of t.

(3) *The short rate is an affine function of the factors, i.e.,*

$$h(t, x) = c(t) + d(t)x,$$

where the scalar c and the $1 \times n$ row vector d are deterministic functions of t.

Furthermore, under the conditions above the functions A and B in (13) *are determined by the following system of ODEs, where the subscript t denotes partial derivative with respect to t, and where D denotes the block matrix $D = [D_1, \ldots, D_n]$:*

$$B_t(t, T) = -B(t, T)\Delta(t) + \frac{1}{2}B(t, T)D(t)B^{\star}(t, T) - d(t), \tag{14}$$

$$B(T, T) = 0. \tag{15}$$

$$A_t(t, T) = B(t, T)\alpha(t, T) - \frac{1}{2}B(t, T)C(t)B^{\star}(t, T) + c(t), \tag{16}$$

$$A(T, T) = 0. \tag{17}$$

Proof. The proof is surprisingly simple. Given the *Ansatz* (13), and the sufficient conditions above, compute the partial derivatives and plug them into the term structure equation. The PDE will then be separable in x and the ODEs are obtained by identifying coefficients. □

We note that, for every fixed T, (14)–(17) is a coupled system of ODEs in the t-variable. We also see that (14) is a Riccati equation for B, whereas (16)–(17) can be integrated directly, once B is computed.

3.5. Short rate models. The simplest type of a factor model is the one where the factor process X is scalar and coincides with the short rate, i.e., $X_t = r_t$ and $h(x) = x$. Such a model will then have the form

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t,$$

where W is Q-Wiener. As we saw in the previous section, the term structure (i.e., the family of bond price processes) will, together with all other derivatives, be completely determined by the term structure equation

$$\frac{\partial F}{\partial t}(t, r) + \mu(t, r)\frac{\partial F}{\partial r}(t, r) + \frac{1}{2}\sigma^2(t, r)\frac{\partial^2 F}{\partial r^2}(t, r) - rF(t, r) = 0, \tag{18}$$

$$F(T, r) = \Phi(r). \tag{19}$$

In the literature there are a large number of proposals on how to specify the Q-dynamics for r. We present a (far from complete) list of the most popular models.

If a parameter is time dependent this is written out explicitly. Otherwise all parameters are constant and positive.

(1) Vasiček: $dr_t = (b - ar_t)\,dt + \sigma\,dW_t$,

(2) Cox–Ingersoll–Ross (CIR): $dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t$,

(3) Dothan: $dr_t = ar_t\,dt + \sigma r_t\,dW_t$,

(4) Black–Derman–Toy (BDT): $dr_t = \Theta(t)r_t\,dt + \sigma(t)r_t\,dW_t$,

(5) Ho–Lee: $dr_t = \Theta(t)\,dt + \sigma\,dW_t$,

(6) Hull–White (extended Vasiček): $dr_t = \{\Theta(t) - a(t)r_t\}\,dt + \sigma(t)\,dW_t$,

(7) Hull–White (extended CIR): $dr_t = \{\Theta(t) - a(t)r\}\,dt + \sigma(t)\sqrt{r_t}\,dW_t$.

3.6. Inverting the yield curve. We now turn to the problem of parameter estimation in the martingale models above, and a natural procedure would perhaps be to use standard statistical estimation procedures based on time series data of the underlying factor process. This procedure, however, is unfortunately completely nonsensical and the reason is as follows.

Let us for simplicity assume we have a short rate model. Now, we have chosen to model the r-process by giving the Q-dynamics, which means that all parameters are defined under the martingale measure Q. When we make observations in the real world we are however *not* observing r under the martingale measure Q, but under the objective measure P. This means that if we apply standard statistical procedures to our observed data we will not get our Q-parameters. What we get instead is pure nonsense.

To see how we can save the situation, we begin by recalling from Result 3.2 that *the martingale measure is chosen by the market*. Thus, in order to obtain information about the Q-drift parameters we have to *collect price information from the market*, and the typical approach is that of *inverting the yield curve* which works as follows.

- Choose a particular short rate model involving one or several parameters. (The arguments below will in fact apply to any factor model, but for simplicity we confine ourselves to short rate models.) Let us denote the entire parameter vector by α. Thus we write the r-dynamics (under Q) as

$$dr_t = \mu(t, r_t; \alpha)\,dt + \sigma(t, r_t; \alpha)\,dW_t. \tag{20}$$

- Solve the term structure equation (18)–(19) to obtain the theoretical term structure as

$$p(t, T; \alpha) = F^T(t, r; \alpha).$$

- Collect price data (at $t = 0$) from the bond market for all maturities. Denote this *empirically observed term structure* by $\{p^\circ(0,T); T \geq 0\}$.
- Now choose the parameter vector α in such a way that the theoretical curve $\{p(0,T;\alpha); T \geq 0\}$ fits the empirical curve $\{p^\circ(0,T); T \geq 0\}$ as well as possible (according to some objective function). This gives us our estimated parameter vector α°.
- We have now determined our martingale measure Q, and we can go on to compute prices of interest rate derivatives.

The procedure above is known as "inverting the yield curve", "backing out parameters from market data", or "calibrating the model to market data".

We end this section by noting that if we want a complete fit between the theoretical and the observed bond prices this calibration procedure is formally that of solving the system of equations

$$p(0,T;\alpha) = p^\circ(0,T) \qquad \text{for all } T > 0. \tag{21}$$

We observe that this is an infinite-dimensional system of equations (one equation for each T) with α as the unknown, so if we work with a model containing a finite parameter vector α (like the Vasiček model) there is no hope of obtaining a perfect fit.

This is the reason why in the Hull–White model we introduce the infinite-dimensional parameter vector Θ and it can in fact be shown that there exists a unique solution to (21) for the Ho–Lee model as well as for Hull–White extended Vasiček and CIR models. As an example, for the Ho–Lee model Θ is given by

$$\Theta(t) = f_T^\circ(0,t) + \sigma^2 t,$$

where the lower index denotes partial derivative with respect to maturity.

It should, however, be noted that the introduction of an infinite parameter, in order to fit the entire initial term structure, has its dangers in terms of over-parameterization, leading to numerical instability of the parameter estimates.

4. Forward rate models and the geometric view

Up to this point we have studied interest models generated by a finite number of underlying factors. The method proposed by Heath–Jarrow–Morton (HJM) is at the far end of this spectrum—they choose the entire forward rate curve as their (infinite-dimensional) state variable.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 333)

4.1. The HJM drift condition. We now turn to the specification of the Heath–Jarrow–Morton framework. This can be done under P or Q, but here we confine ourselves to Q modeling.

Assumption 4.1. We assume that, for every fixed $T > 0$, the forward rate $f(\cdot, T)$ has a stochastic differential which, under a given martingale measure Q, is given by

$$df(t, T) = \alpha(t, T)\,dt + \sigma(t, T)\,dW(t), \tag{22}$$

$$f(0, T) = f^{\circ}(0, T), \tag{23}$$

where W is a (d-dimensional) Q-Wiener process whereas $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes.

Note that conceptually equation (22) is a scalar stochastic differential in the t-variable for each fixed choice of T. The index T thus only serves as a "mark" or "parameter" in order to indicate which maturity we are looking at. Also note that we use the observed forward rated curve $\{f^{\circ}(0, T); T \geq 0\}$ as the initial condition. This will automatically give us a perfect fit between observed and theoretical bond prices at $t = 0$, thus relieving us of the task of inverting the yield curve.

Remark 4.2. It is important to observe that the HJM approach to interest rates does not a propose of a specific *model*, like, for example, the Vasiček model. It is instead a *framework* to be used for analyzing interest rate models. We do not have a specific model until we have specified the drift and volatility structure in (22). Every short rate model can be equivalently formulated in forward rate terms, and for every forward rate model, the arbitrage free price of a contingent T-claim Y will still be given by the pricing formula

$$\Pi(0; Y) = E^{Q}[e^{-\int_0^T r(s)\,ds} \cdot Y],$$

where the short rate as usual is given by $r(s) = f(s, s)$.

We noticed earlier that for a short rate model *every* $Q \sim P$ will serve as a martingale measure. This is not the case for a forward rate model, the reason being that we have the following two different formulas for bond prices

$$p(t, T) = e^{-\int_t^T f(t,s)\,ds},$$

$$p(t, T) = E^{Q}[e^{-\int_0^T r(s)\,ds} \mid \mathcal{F}_t],$$

where the short rate r and the forward rate f are connected by $r(t) = f(t, t)$. In order for these formulas to hold simultaneously, we have to impose some sort of consistency relation between α and σ in the forward rate dynamics. The result is the famous Heath–Jarrow–Morton drift condition.

Proposition 4.3 (HJM drift condition). *Under the martingale measure Q, the processes α and σ must satisfy the following relation, for every t and every $T \geq t$.*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^\star\,ds. \tag{24}$$

Proof. From Proposition 2.5 we obtain the bond price dynamics as

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt + p(t,T)S(t,T)\,dW(t).$$

We also know that, under a martingale measure, the local rate of return has to equal the short rate r. Thus we obtain the identity

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 = 0,$$

and differentiating this with respect to T gives us (24). □

The moral of Proposition 4.3 is that when we specify the forward rate dynamics (under Q) we may freely specify the volatility structure. The drift parameters are then uniquely determined.

4.2. The Musiela parameterization. In many practical applications it is more natural to use time *to* maturity, rather than time *of* maturity, to parameterize bonds and forward rates. If we denote running time by t, time of maturity by T, and time to maturity by x, then we have $x = T - t$, and in terms of x the forward rates are defined as follows.

Definition 4.4. For all $x \geq 0$ the forward rates $r(t,x)$ are defined by the relation

$$r(t,x) = f(t,t+x).$$

Suppose now that we have the standard HJM-type model for the forward rates under a martingale measure Q

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t). \tag{25}$$

The question is to find the Q-dynamics for $r(t,x)$, and we have the following result, known as the Musiela equation.

Proposition 4.5 (The Musiela equation). *Assume that the forward rate dynamics under Q are given by* (25). *Then*

$$dr(t,x) = \left\{ \frac{\partial}{\partial x} r(t,x) + D(t,x) \right\} dt + \sigma_0(t,x)\, dW(t), \tag{26}$$

where

$$\sigma_0(t,x) = \sigma(t, t+x),$$
$$D(t,x) = \sigma_0(t,x) \int_0^x \sigma_0(t,s)^\star \, ds.$$

Proof. Using a slight variation of the Itô formula we have

$$dr(t,x) = df(t, t+x) + \frac{\partial f}{\partial T}(t, t+x)\, dt,$$

where the differential in the term $df(t, t+x)$ only operates on the first t. We thus obtain

$$dr(t,x) = \alpha(t, t+x)\, dt + \sigma(t, t+x)\, dW(t) + \frac{\partial}{\partial x} r(t,x)\, dt,$$

and, using the HJM drift condition, we obtain our result. □

The point of the Musiela parameterization is that it highlights equation (26) as an infinite-dimensional SDE. It has become an indispensable tool of modern interest rate theory.

In the remaining of this section we give briefly overview the geometric view on forward rate models. This way of thinking about term structures was first proposed by Björk and Christensen [3] and Björk and Svensson [8].

4.3. The geometric setup and main problems. We consider a given forward rate model under a risk neutral martingale measure Q. We will adopt the Musiela parameterization and thus use the notation $r(t,x) = f(t, t+x)$. Recall from proposition 4.5 that under the martingale measure Q the r-dynamics are given by

$$dr(t,x) = \left\{ \frac{\partial}{\partial x} r(t,x) + \sigma(t,x) \int_0^x \sigma(t,u)^\star \, du \right\} dt + \sigma(t,x)\, dW(t), \tag{27}$$

$$r(0,x) = r^o(0,x). \tag{28}$$

where, as before, * denotes transpose and r^o is the initial forward rate curve.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 336)

Note that from now on r denotes forwards rates. To avoid confusion, we also change the notation of short rate to R (i.e., $R(t) = r(t, 0)$).

Suppose that we are given a concrete model $\mathscr{M}$ within the above framework, i.e., suppose that we are given a concrete specification of the volatility process σ. We then have some *natural problems* to study, and we start from a calibration point of view.

A standard procedure when dealing with concrete interest rate models on a high frequency (say, daily) basis can be described as follows:

(1) At time $t = 0$, use market data to fit (calibrate) the model to the observed bond prices.

(2) Use the calibrated model to compute prices of various interest rate derivatives.

(3) The following day ($t = 1$), repeat the procedure in (1) above in order to recalibrate the model, etc.

To carry out the calibration in step 1. above, the analyst typically has to produce a forward rate curve $\{r^o(0, x); x \geq 0\}$ from the observed data. However, since only a finite number of bonds actually trade in the market, the data consist of a discrete set of points, and a need to fit a curve to these points arises. This curve-fitting may be done in a variety of ways. One way is to use splines, but also a number of parameterized families of smooth forward rate curves have become popular in applications—the most well-known probably being the Nelson-Siegel family (see [43]). Once the curve $\{r^o(0, x); x \geq 0\}$ has been obtained, the parameters of the interest rate model may be calibrated to this.

Now, from a purely logical point of view, the recalibration procedure in step 3. above is of course slightly nonsensical: If the interest rate model at hand is an exact picture of reality, then there should be no need to recalibrate. The reason that everyone insists on recalibrating is of course that any model in fact only is an approximate picture of the financial market under consideration, and recalibration allows incorporating newly arrived information in the approximation. Even so, the calibration procedure itself ought to take into account that it will be repeated. It appears that the optimal way to do so would involve a combination of time series and cross-section data, as opposed to the purely cross-sectional curve-fitting, where the information contained in previous curves is discarded in each recalibration.

The cross-sectional fitting of a forward curve and the repeated recalibration is thus, in a sense, a pragmatic and somewhat non-theoretical endeavor. Nonetheless, there are some nontrivial theoretical problems to be dealt with in this context, and the problem to be studied in this section concerns the *consistency* between, on the one hand, the dynamics of a given interest rate model, and, on the other hand, the forward curve family employed.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 337)

What, then, is meant by consistency in this context? Assume that a given interest rate model $\mathscr{M}$ (e.g., the Hull–White extension of the Vasiček model) in fact *is* an exact picture of the financial market. Now consider a particular family $\mathscr{G}$ of forward rate curves (e.g., the Nelson–Siegel family) and assume that the interest rate model is calibrated using this family. We then say that the pair $\mathscr{M}$ and $\mathscr{G}$ are consistent if all forward curves which can be produced by the interest rate model $\mathscr{M}$ are contained within the family $\mathscr{G}$. Otherwise, the pair $(\mathscr{M}, \mathscr{G})$ is inconsistent.

Thus, if $\mathscr{M}$ and $\mathscr{G}$ are consistent, then the interest rate model actually produces forward curves which belong to the relevant family. In contrast, if $\mathscr{M}$ and $\mathscr{G}$ are inconsistent, then the interest rate model will produce forward curves outside the family used in the calibration step, and this will force the analyst to change the model parameters all the time—not because the model is an approximation to reality, but simply because the family does not go well with the model.

Put into more operational terms this can be rephrased as follows: Suppose that you are using a fixed interest rate model $\mathscr{M}$. If you want to do recalibration, then your family $\mathscr{G}$ of forward rate curves should be chosen is such a way as to be consistent with the model $\mathscr{M}$.

Note however that the argument also can be run backwards, yielding the following conclusion for empirical work.

Suppose that a particular forward curve family $\mathscr{G}$ has been observed to provide a good fit, on a day-to-day basis, in a particular bond market. Then this gives you modeling information about the choice of an interest rate model in the sense that you should try to use/construct an interest rate model which is consistent with the family $\mathscr{G}$.

We can now formulate our main problems:

(1) Under which conditions is a given forward rate model $\mathscr{M}$ and a parameterized family $\mathscr{G}$ of forward rate curves *consistent*?

(2) When can the given, inherently infinite-dimensional, interest rate model $\mathscr{M}$ be written as a *finite dimensional state space model*? More precisely, we seek conditions under which the forward rate process $r(t,x)$ induced by the model $\mathscr{M}$, can be realized by a system of the form

$$dZ_t = a(Z_t)\,dt + b(Z_t)\,dW_t, \tag{29}$$

$$r(t,x) = G(Z_t, x), \tag{30}$$

where Z (interpreted as the state vector process) is a finite dimensional diffusion, $a(z)$, $b(z)$ and $G(z,x)$ are deterministic functions and W is the same Wiener process as in (27).

As will be seen below, these two problems are intimately connected.

4.4. Consistency and invariant manifolds. We start by looking into the first problem and study when a given sub manifold of forward rate curves is consistent (in the sense described above) with a given interest rate model. This problem is of interest from an applied as well as from a theoretical point of view. In particular we will use the results from this section to analyze problems about existence of finite dimensional factor realizations for interest rate models on forward rate form.

We now move on to give precise mathematical definition of the consistency property discussed above, and this leads us to the concept of an *invariant manifold*.

Definition 4.6 (Invariant manifold). Take as given the forward rate process dynamics (27). Consider also a fixed family (manifold) of forward rate curves $\mathscr{G}$. We say that $\mathscr{G}$ is locally *invariant* under the action of r if, for each point $(s, r) \in R_+ \times \mathscr{G}$, the condition $r_s \in \mathscr{G}$ implies that $r_t \in \mathscr{G}$ on a time interval with positive length. If r stays forever on $\mathscr{G}$, we say that $\mathscr{G}$ is globally invariant.

We will characterize invariance in terms of local characteristics of $\mathscr{G}$ and $\mathscr{M}$, and in this context local invariance is the best one can hope for. In order to save space, local invariance will therefore be referred to as invariance.

4.4.1. The formalized problem. As our basic space of forward rate curves we will use a weighted Sobolev space, where a generic point will be denoted by r.

Definition 4.7. Consider a fixed real number $\gamma > 0$. The space $\mathscr{H}_\gamma$ is defined as the space of all differentiable (in the distributional sense) functions

$$r : R_+ \to R$$

satisfying the norm condition $\|r\|_\gamma < \infty$. Here the norm is defined as

$$\|r\|_\gamma^2 = \int_0^\infty r^2(x) e^{-\gamma x}\, dx + \int_0^\infty \left(\frac{dr}{dx}(x) \right)^2 e^{-\gamma x}\, dx.$$

Remark 4.8. The variable x is as before interpreted as time to maturity. With the inner product

$$(r, q) = \int_0^\infty r(x) q(x) e^{-ax}\, dx + \int_0^\infty \left(\frac{dr}{dx}(x) \right) \left(\frac{dq}{dx}(x) \right) e^{-\gamma x}\, dx,$$

the space $\mathscr{H}_\gamma$ becomes a Hilbert space. Because of the exponential weighting function all constant forward rate curves will belong to the space. In the sequel we will suppress the subindex γ, writing $\mathscr{H}$ instead of $\mathscr{H}_\gamma$.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 339)

4.4.2. The forward curve manifold. We consider as given a mapping

$$G : \mathscr{Z} \to \mathscr{H}, \tag{31}$$

where the parameter space $\mathscr{Z}$ is an open connected subset of R^d, i.e., for each parameter value $z \in \mathscr{Z} \subseteq R^d$ we have a curve $G(z) \in \mathscr{H}$. The value of this curve at the point $x \in R_+$ will be written as $G(z, x)$, so we see that G can also be viewed as a mapping

$$G : \mathscr{Z} \times R_+ \to R. \tag{32}$$

The mapping G is thus a formalization of the idea of a finitely parameterized family of forward rate curves, and we now define the forward curve manifold as the set of all forward rate curves produced by this family.

Definition 4.9. The *forward curve manifold* $\mathscr{G} \subseteq \mathscr{H}$ is defined as

$$\mathscr{G} = \mathrm{Im}(G).$$

4.4.3. The interest rate model. We take as given a volatility function σ of the form

$$\sigma : \mathscr{H} \times R_+ \to R^m$$

i.e., $\sigma(r, x)$ is a functional of the infinite-dimensional r-variable, and a function of the real variable x. Denoting the forward rate curve at time t by r_t we then have the following forward rate equation.

$$dr_t(x) = \left\{\frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x)\int_0^x \sigma(r_t, u)^\star \, du\right\} dt + \sigma(r_t, x)\, dW_t. \tag{33}$$

Remark 4.10. For notational simplicity we have assumed that the r-dynamics are time homogeneous. The case when σ is of the form $\sigma(t, r, x)$ can be treated in exactly the same way (see [3]).

We also need some regularity assumptions, but we suppress these here and refer to [3] for technical details.

4.4.4. The invariance conditions. In order to study the invariance problem we need to use some compact notation.

Definition 4.11. We define $\boldsymbol{H}\sigma$ by

$$\boldsymbol{H}\sigma(r, x) = \int_0^x \sigma(r, s)\, ds.$$

Suppressing the x-variable, the Itô dynamics for the forward rates are thus given by

$$dr_t = \left\{ \frac{\partial}{\partial x} r_t + \sigma(r_t) \boldsymbol{H} \sigma(r_t)^\star \right\} dt + \sigma(r_t)\, dW_t, \tag{34}$$

and we write this more compactly as

$$dr_t = \mu_0(r_t)\, dt + \sigma(r_t)\, dW_t, \tag{35}$$

where the drift μ_0 is given by the bracket term in (34). To get some intuition we now formally "divide by dt" and obtain

$$\frac{dr}{dt} = \mu_0(r_t) + \sigma(r_t)\, \dot{W}_t, \tag{36}$$

where the formal time derivative $\dot{W}_t$ is interpreted as an "input signal" chosen by chance. We are thus led to study the associated deterministic control system

$$\frac{dr}{dt} = \mu_0(r_t) + \sigma(r_t) u_t. \tag{37}$$

The intuitive idea is now that $\mathscr{G}$ is invariant under (35) if and only if $\mathscr{G}$ is invariant under (37) for all choices of the input signal u. It is furthermore geometrically obvious that this happens if and only if the velocity vector $\mu(r) + \sigma(r)u$ is tangential to $\mathscr{G}$ for all points $r \in \mathscr{G}$ and all choices of $u \in R^m$. Since the tangent space of $\mathscr{G}$ at a point $G(z)$ is given by $\mathrm{Im}[G_z'(z)]$, where G_z' denotes the Frechet derivative (Jacobian), we are led to conjecture that $\mathscr{G}$ is invariant if and only if the condition

$$\mu_0(r) + \sigma(r) u \in \mathrm{Im}[G_z'(z)]$$

is satisfied for all $u \in R^m$. This can also be written

$$\mu_0(r) \in \mathrm{Im}[G_z'(z)], \qquad \sigma(r) \in \mathrm{Im}[G_z'(z)],$$

where the last inclusion is interpreted componentwise for σ.

This "result" is, however, not correct due to the fact that the argument above neglects the difference between ordinary calculus, which is used for (37), and Itô calculus, which governs (35). In order to bridge this gap we have to rewrite the analysis in terms of Stratonovich integrals instead of Itô integrals.

Definition 4.12. For given semimartingales X and Y, the *Stratonovich integral* $\int_0^t X(s) \circ dY(s)$ of X with respect to Y is defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s\, dY_s + \frac{1}{2} \langle X, Y \rangle_t. \tag{38}$$

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 341)

The first term on the RHS is the Itô integral. In the present case, with only Wiener processes as driving noise, we can define the 'quadratic variation process' $\langle X, Y \rangle$ in (38) by

$$d\langle X, Y \rangle_t = dX_t \, dY_t, \tag{39}$$

with the usual 'multiplication rules' $dW \cdot dt = dt \cdot dt = 0$, $dW \cdot dW = dt$. We now recall the main result and *raison d'être* for the Stratonovich integral.

Proposition 4.13 (Chain rule). *Assume that the function $F(t, y)$ is smooth. Then we have*

$$dF(t, Y_t) = \frac{\partial F}{\partial t}(t, Y_t) \, dt + \frac{\partial F}{\partial y} \circ dY_t. \tag{40}$$

Thus, in the Stratonovich calculus, the Itô formula takes the form of the standard chain rule of ordinary calculus.

Returning to (35), the Stratonovich dynamics are given by

$$dr_t = \left\{ \frac{\partial}{\partial x} r_t + \sigma(r_t) \boldsymbol{H} \sigma(r_t)^{\star} \right\} dt - \frac{1}{2} d\langle \sigma(r_t), W_t \rangle + \sigma(r_t) \circ dW_t. \tag{41}$$

In order to compute the Stratonovich correction term above we use the infinite-dimensional Itô formula (see [16]) to obtain

$$d\sigma(r_t) = \{\ldots\} \, dt + \sigma_r'(r_t) \sigma(r_t) \, dW_t, \tag{42}$$

where σ_r' denotes the Frechet derivative of σ with respect to the infinite-dimensional r-variable. From this we immediately obtain

$$d\langle \sigma(r_t), W_t \rangle = \sigma_r'(r_t) \sigma(r_t) \, dt. \tag{43}$$

Remark 4.14. If the Wiener process W is multidimensional, then σ is a vector $\sigma = [\sigma_1, \ldots, \sigma_m]$, and the right-hand side of (43) should be interpreted as

$$\sigma_r'(r_t) \sigma(r_t) = \sum_{i=1}^{m} \sigma_{ir}'(r_t) \sigma_i(r_t).$$

Thus (41) becomes

$$dr_t = \left\{ \frac{\partial}{\partial x} r_t + \sigma(r_t) \boldsymbol{H} \sigma(r_t)^{\star} - \frac{1}{2} \sigma_r'(r_t) \sigma(r_t) \right\} dt + \sigma(r_t) \circ dW_t \tag{44}$$

We now write (44) as

$$dr_t = \mu(r_t)\,dt + \sigma(r_t) \circ dW_t, \tag{45}$$

where

$$\mu(r,x) = \frac{\partial}{\partial x} r(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, u)^\star \, du - \frac{1}{2}[\sigma_r'(r_t)\sigma(r_t)](x). \tag{46}$$

Given the heuristics above, our main result is not surprising. The formal proof, which is somewhat technical, is left out. See [3].

Theorem 4.15 (Main Theorem). *The forward curve manifold $\mathcal{G}$ is locally invariant for the forward rate process $r(t,x)$ in $\mathcal{M}$ if and only if*

$$G_x'(z) + \sigma(r)\boldsymbol{H}\sigma(r)^\star - \frac{1}{2}\sigma_r'(r)\sigma(r) \in \mathrm{Im}[G_z'(z)], \tag{47}$$

$$\sigma(r) \in \mathrm{Im}[G_z'(z)], \tag{48}$$

hold for all $z \in \mathcal{Z}$ with $r = G(z)$.

Here, G_z' and G_x' denote the Frechet derivative of G with respect to z and x, respectively. The condition (48) is interpreted componentwise for σ. Condition (47) is called *the consistent drift condition*, and (48) is called *the consistent volatility condition*.

Remark 4.16. It is easily seen that if the family G is invariant under shifts in the x-variable, then we will automatically have the relation

$$G_x'(z) \in \mathrm{Im}[G_z'(z)],$$

so in this case the relation (47) can be replaced by

$$\sigma(r)\boldsymbol{H}\sigma(r)^\star - \frac{1}{2}\sigma_r'(r)\sigma(r) \in \mathrm{Im}[G_z'(z)],$$

with $r = G(z)$ as usual.

The results above are extremely easy to apply in concrete situations. As a test case we consider the Nelson–Siegel family of forward rate curves. We analyze the consistency of this family with the Hull–White extension of the Vasiček model.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 343)

4.4.5. Example: the NS family and the HW extended Vasiček model. The Nelson–Siegel (henceforth NS) forward curve manifold $\mathcal{G}$ is parameterized by $z \in R^4$, the curve $x \mapsto G(z, x)$ as

$$G(z, x) = z_1 + z_2 e^{-z_4 x} + z_3 x e^{-z_4 x} \,. \tag{49}$$

For $z_4 \neq 0$, the Frechet derivatives are easily obtained as

$$G_z'(z, x) = [1, e^{-z_4 x}, x e^{-z_4 x}, -(z_2 + z_3 x) x e^{-z_4 x}], \tag{50}$$

$$G_x'(z, x) = (z_3 - z_2 z_4 - z_3 z_4 x) e^{-z_4 x}. \tag{51}$$

In the degenerate case $z_4 = 0$, we have

$$G(z, x) = z_1 + z_2 + z_3 x, \tag{52}$$

We return to this case below.

As our test case, we analyze the Hull and White (henceforth HW) extension of the Vasiček model. On short rate form the model is given by

$$dR(t) = \{\Phi(t) - aR(t)\}\, dt + \sigma\, dW(t), \tag{53}$$

where $a, \sigma > 0$. As is well known, the corresponding forward rate formulation is

$$dr(t, x) = \beta(t, x)\, dt + \sigma e^{-ax}\, dW_t. \tag{54}$$

Thus, the volatility function is given by $\sigma(x) = \sigma e^{-ax}$, and the conditions of Theorem 4.15 become

$$G_x'(z, x) + \frac{\sigma^2}{a}[e^{-ax} - e^{-2ax}] \in \mathrm{Im}[G_z'(z, x)], \tag{55}$$

$$\sigma e^{-ax} \in \mathrm{Im}[G_z'(z, x)]. \tag{56}$$

To investigate whether the NS manifold is invariant under HW dynamics, we start with (56) and fix a z-vector. We then look for constants (possibly depending on z) A, B, C and D, such that for all $x \geq 0$ we have

$$\sigma e^{-ax} = A + B e^{-z_4 x} + C x e^{-z_4 x} - D(z_2 + z_3 x) x e^{-z_4 x}. \tag{57}$$

This is possible if and only if $z_4 = a$, and since (56) must hold for all choices of $z \in \mathcal{Z}$ we immediately see that HW is inconsistent with the full NS manifold. See [24] for a remarkable extension of this result.

Proposition 4.17 (Nelson–Siegel and Hull–White). *The Hull–White model is inconsistent with the NS family.*

We have thus obtained a negative result for the HW model. The NS manifold is 'too small' for HW, in the sense that if the initial forward rate curve is on the manifold, then the HW dynamics will force the term structure off the manifold within an arbitrarily short period of time. For more positive results see [3].

Remark 4.18. It is an easy exercise to see that the minimal manifold which is consistent with HW is given by

$$G(z, x) = z_1 e^{-ax} + z_2 e^{-2ax}.$$

4.5. Existence of finite realizations. We now turn to Problem 2 in Section 4.3, i.e., the problem when a given forward rate model has a finite dimensional factor realization. For ease of exposition we mostly confine ourselves to a discussion of the case of a single driving Wiener process and to time invariant forward rate dynamics. Multidimensional Wiener processes and time varying systems can be treated similarly, and for completeness we state the results for the multidimensional case. We will use some ideas and concepts from differential geometry, and a general reference here is [48].

We now take as given a volatility $\sigma : \mathscr{H} \to \mathscr{H}$ and consider the induced forward rate model (on Stratonovich form)

$$dr_t = \mu(r_t)\, dt + \sigma(r_t) \circ dW_t, \tag{58}$$

where as before (see Section 4.4.4)

$$\mu(r) = \frac{\partial}{\partial x} r + \sigma(r) \boldsymbol{H} \sigma(r)^{\star} - \frac{1}{2} \sigma_r'(r) \sigma(r). \tag{59}$$

Remark 4.19. The reason for our choice of $\mathscr{H}$ as the underlying space, is that the linear operator $\mathbf{F} = d/dx$ is bounded in this space. Together with the assumptions above, this implies that both μ and σ are smooth vector fields on $\mathscr{H}$, thus ensuring the existence of a strong local solution to the forward rate equation for every initial point $r^o \in \mathscr{H}$.

4.5.1. The geometric problem. Given a specification of the volatility mapping σ, and an initial forward rate curve r^o we now investigate when (and how) the corresponding forward rate process possesses a finite, dimensional realization. We are thus looking for smooth d-dimensional vector fields a and b, an initial point $z_0 \in R^d$, and a mapping $G : R^d \to \mathscr{H}$ such that r, locally in time, has the representation

$$dZ_t = a(Z_t)\, dt + b(Z_t)\, dW_t, \qquad Z_0 = z_0, \tag{60}$$

$$r(t, x) = G(Z_t, x). \tag{61}$$

Remark 4.20. Let us clarify some points. Firstly, note that in principle it may well happen that, given a specification of σ, the r-model has a finite dimensional realization given a particular initial forward rate curve r^o, while being infinite-dimensional for all other initial forward rate curves in a neighbourhood of r^o. We say that such a model is a *non-generic* or *accidental* finite dimensional model. If, on the other hand, r has a finite dimensional realization for all initial points in a neighbourhood of r^o, then we say that the model is a *generically* finite dimensional model. In this text we are solely concerned with the generic problem. Secondly, let us emphasize that we are looking for *local* (in time) realizations.

We can now connect the realization problem to our studies of invariant manifolds.

Proposition 4.21. *The forward rate process possesses a finite dimensional realization if and only if there exists an invariant finite dimensional submanifold $\mathcal{G}$ with $r^o \in \mathcal{G}$.*

Proof. See [3] for the full proof. The intuitive argument runs as follows. Suppose that there exists a finite dimensional invariant manifold $\mathcal{G}$ with $r^o \in \mathcal{G}$. Then $\mathcal{G}$ has a local coordinate system, and we may define the Z process as the local coordinate process for the r-process. On the other hand it is clear that if r has a finite dimensional realization as in (60)–(61), then every forward rate curve that will be produced by the model is of the form $x \mapsto G(z, x)$ for some choice of z. Thus there exists a finite dimensional invariant submanifold $\mathcal{G}$ containing the initial forward rate curve r^o, namely $\mathcal{G} = \mathrm{Im}(G)$. □

Using Theorem 4.15 we immediately obtain the following geometric characterisation of the existence of a finite realization.

Corollary 4.22. *The forward rate process possesses a finite dimensional realization if and only if there exists a finite dimensional manifold $\mathcal{G}$ containing r^o, such that, for each $r \in \mathcal{G}$ the following conditions hold:*

$$\mu(r) \in T_{\mathcal{G}}(r), \qquad \sigma(r) \in T_{\mathcal{G}}(r).$$

Here $T_{\mathcal{G}}(r)$ denotes the tangent space to $\mathcal{G}$ at the point r, and the vector fields μ and σ are as above.

4.5.2. The main result. Given the volatility vector field σ, and hence also the field μ, we now are faced with the problem of determining if there exists a finite dimensional manifold $\mathcal{G}$ with the property that μ and σ are tangential to $\mathcal{G}$ at each point of $\mathcal{G}$. In the case when the underlying space is finite dimensional, this is a standard problem in differential geometry, and we will now give the heuristics.

To get some intuition we start with a simpler problem and therefore consider the space $\mathscr{H}$ (or any other Hilbert space), and a smooth vector field f on the space. For each fixed point $r^o \in \mathscr{H}$ we now ask if there exists a finite dimensional manifold $\mathscr{G}$ with $r^o \in \mathscr{G}$ such that f is tangential to $\mathscr{G}$ at every point. The answer to this question is yes, and the manifold can in fact be chosen to be one-dimensional. To see this, consider the infinite-dimensional ODE

$$\frac{dr_t}{dt} = f(r_t), \tag{62}$$

$$r_0 = r^o. \tag{63}$$

If r_t is the solution, at time t, of this ODE, we use the notation

$$r_t = e^{ft} r^o.$$

We have thus defined a group of operators $\{e^{ft} : t \in R\}$, and we note that the set $\{e^{ft} r^o : t \in R\} \subseteq \mathscr{H}$ is nothing else than the integral curve of the vector field f, passing through r^o. If we define $\mathscr{G}$ as this integral curve, then our problem is solved, since f will be tangential to $\mathscr{G}$ by construction.

Let us now take two vector fields f_1 and f_2 as given, where the reader informally can think of f_1 as σ and f_2 as μ. We also fix an initial point $r^o \in \mathscr{H}$ and the question is if there exists a finite dimensional manifold $\mathscr{G}$, containing r^o, with the property that f_1 and f_2 are both tangential to $\mathscr{G}$ at each point of $\mathscr{G}$. We call such a manifold an *tangential manifold* for the vector fields. At a first glance it would seem that there always exists an tangential manifold, and that it can even be chosen to be two-dimensional. The geometric idea is that we start at r^o and let f_1 generate the integral curve $\{e^{f_1 s} r^o : s \geq 0\}$. For each point $e^{f_1 s} r^o$ on this curve we now let f_2 generate the integral curve starting at that point. This gives us the object $e^{f_2 t} e^{f_1 s} r^o$ and thus it seems that we sweep out a two dimensional surface $\mathscr{G}$ in $\mathscr{H}$. This is our obvious candidate for an tangential manifold.

In the general case this idea will, however, not work, and the basic problem is as follows. In the construction above we started with the integral curve generated by f_1 and then applied f_2, and there is of course no guarantee that we will obtain the same surface if we start with f_2 and then apply f_1. We thus have some sort of commutativity problem, and the key concept is the *Lie bracket*.

Definition 4.23. Given smooth vector fields f and g on $\mathscr{H}$, the Lie bracket $[f, g]$ is a new vector field defined by

$$[f, g](r) = f'(r)g(r) - g'(r)f(r). \tag{64}$$

The Lie bracket measures the lack of commutativity on the infinitesimal scale in our geometric program above, and for the procedure to work we need a condi-

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 347)

tion which says that the lack of commutativity is "small". It turns out that the relevant condition is that the Lie bracket should be in the linear hull of the vector fields.

Definition 4.24. Let $f_1, \ldots, f_n$ be smooth independent vector fields on some space X. Such a system is called a *distribution*, and the distribution is said to be *involutive* if

$$[f_i, f_j](x) \in \operatorname{span}\{f_1(x), \ldots, f_n(x)\} \qquad \text{for all } i, j,$$

where the span is the linear hull over the real numbers.

We now have the following basic result, which extends a classic result from finite dimensional differential geometry (see [48]).

Theorem 4.25 (Frobenius). *Let $f_1, \ldots, f_k$ and be linearly independent smooth vector fields in $\mathscr{H}$ and consider a fixed point $r^o \in \mathscr{H}$. Then the following statements are equivalent.*

- *For each point r in a neighbourhood of r^o, there exists a k-dimensional tangential manifold passing through r.*
- *The system $f_1, \ldots, f_k$ of vector fields is (locally) involutive.*

Proof. See [8], which provides a self-contained proof of the Frobenius Theorem in Banach space. □

Let us now go back to our interest rate model. We are thus given the vector fields μ, σ, and an initial point r^o, and the problem is whether there exists a finite dimensional tangential manifold containing r^o. Using the infinite-dimensional Frobenius theorem, this situation is now easily analyzed. If $\{\mu, \sigma\}$ is involutive then there exists a two dimensional tangential manifold. If $\{\mu, \sigma\}$ is not involutive, this means that the Lie bracket $[\mu, \sigma]$ is not in the linear span of μ and σ, so we then consider the system $\{\mu, \sigma, [\mu, \sigma]\}$. If this system is involutive there exists a three dimensional tangential manifold. If it is not involutive at least one of the brackets $[\mu, [\mu, \sigma]]$, $[\sigma, [\mu, \sigma]]$ is not in the span of $\{\mu, \sigma, [\mu, \sigma]\}$, and we then adjoin this (these) bracket(s). We continue in this way, forming brackets of brackets, and adjoining these to the linear hull of the previously obtained vector fields, until the point when the system of vector fields thus obtained actually is closed under the Lie bracket operation.

Definition 4.26. Take the vector fields $f_1, \ldots, f_k$ as given. The *Lie algebra* generated by $f_1, \ldots, f_k$ is the smallest linear space (over $\mathbb{R}$) of vector fields which con-

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 348)

tains $f_1, \dots, f_k$ and is closed under the Lie bracket. This Lie algebra is denoted by

$$\mathscr{L} = \{f_1, \dots, f_k\}_{\mathrm{LA}}$$

The *dimension* of $\mathscr{L}$ is defined, for each point $r \in \mathscr{H}$, as

$$\dim[\mathscr{L}(r)] = \dim \operatorname{span}\{f_1(r), \dots, f_k(r)\}_{\mathrm{LA}}.$$

Putting all these results together, we have the following main result on finite dimensional realizations.

Theorem 4.27 (Main result). *Take the volatility mapping $\sigma = (\sigma_1, \dots, \sigma_m)$ as given. Then the forward rate model generated by σ generically admits a finite dimensional realization if and only if*

$$\dim\{\mu, \sigma_1, \dots, \sigma_m\}_{\mathrm{LA}} < \infty$$

in a neighbourhood of r^o.

When computing the Lie algebra generated by μ and σ, the following observations are often useful.

Lemma 4.28. *Take the vector fields $f_1, \dots, f_k$ as given. The Lie algebra $\mathscr{L} = \{f_1, \dots, f_k\}_{\mathrm{LA}}$ remains unchanged under the following operations.*

- *The vector field $f_i(r)$ may be replaced by $\alpha(r) f_i(r)$, where α is any smooth nonzero scalar field.*
- *The vector field $f_i(r)$ may be replaced by*

$$f_i(r) + \sum_{j \neq i} \alpha_j(r) f_j(r),$$

where α_j is any smooth scalar field.

Proof. The first point is geometrically obvious, since multiplication by a scalar field will only change the length of the vector field f_i, and not its direction, and thus not the tangential manifold. Formally it follows from the "Leibnitz rule" $[f, \alpha g] = \alpha[f, g] - (\alpha' f) g$. The second point follows from the bilinear property of the Lie bracket together with the fact that $[f, f] = 0$. □

We conclude this general section by pointing out that although the Lie algebra approach described above allows us to completely solve the FDR problem, it has a serious limitation in the sense that it relies heavily on the assumption that the

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 349)

driving processes are Wiener processes. If you introduce a driving point process in the dynamics of the forward rates, then the Lie algebra approach cannot be used in a straightforward manner. The reason is basically that the Lie algebra approach relies on *local* analysis, such as differential calculus and the Frobenius Theorem. Local analysis is well suited to Wiener framework since a Wiener process acts locally in time and space, and this is reflected by the fact that the infinitesimal operator is a differential operator. A point process, on the other hand, exhibits *global* behavior by forcing the forward rate curve to jump from one point in the space to another and this is reflected in the fact that the infinitesimal operator is an integral operator. For a general theory including point processes we thus need completely new arguments, and one interesting step in this direction is taken in the PhD thesis [46]. For point processes driving only the volatility, but not entering directly into the forward rate dynamcis, the situation is simpler, and the Lie algebra approach can be used. See [22].

4.5.3. Application 1: the case of constant volatility. We now present some simple applications of the theory developed above, but first we need to recall some facts about quasi-exponential functions.

Definition 4.29. A *quasi-exponential* (or QE) function is by definition any function of the form

$$f(x) = \sum_u e^{\lambda_u x} + \sum_j e^{\alpha_j x}[p_j(x)\cos(w_j x) + q_j(x)\sin(w_j x)],$$

where λ_u, α_j, w_j are real numbers, whereas p_j and q_j are real polynomials.

QE functions will turn up over and over again, so we list some simple well known property.

Lemma 4.30. *A function is QE if and only if it is a component of the solution of a linear ODE with constant coefficients.*

We start with the simplest case, which is when the volatility $\sigma(r,x)$ is a constant vector in $\mathscr{H}$, and we assume for simplicity that we have only one driving Wiener process. Then we have no Stratonovich correction term and the vector fields are given by

$$\mu(r,x) = \mathbf{F}r(x) + \sigma(x)\int_0^x \sigma(s)\,ds, \qquad \sigma(r,x) = \sigma(x).$$

where $\mathbf{F} = \frac{\partial}{\partial x}$.

The Frechet derivatives are trivial in this case. Since $\mathbf{F}$ is linear (and bounded in our space), and σ is constant as a function of r, we obtain

$$\mu_r' = \mathbf{F}, \qquad \sigma_r' = 0.$$

Thus the Lie bracket $[\mu, \sigma]$ is given by

$$[\mu, \sigma] = \mathbf{F}\sigma,$$

and in the same way we have

$$[\mu, [\mu, \sigma]] = \mathbf{F}^2\sigma.$$

Continuing in the same manner it is easily seen that the relevant Lie algebra $\mathscr{L}$ is given by

$$\mathscr{L} = \{\mu, \sigma\}_{\mathrm{LA}} = \mathrm{span}\{\mu, \sigma, \mathbf{F}\sigma, \mathbf{F}^2\sigma, \ldots\} = \mathrm{span}\{\mu, \mathbf{F}^n\sigma; n = 0, 1, 2, \ldots\}.$$

It is thus clear that $\mathscr{L}$ is finite dimensional (at each point r) if and only if the function space

$$\mathrm{span}\{\mathbf{F}^n\sigma; n = 0, 1, 2, \ldots\}$$

is finite dimensional. We have thus obtained the following result.

Proposition 4.31. *Under the above assumptions, there exists a finite dimensional realization if and only if σ is a quasi-exponential function.*

4.5.4. Application 2: the case of constant direction volatility. We go on to study the most natural extension of the deterministic volatility case (still in the case of a scalar Wiener process) namely the case when the volatility is of the form

$$\sigma(r, x) = \varphi(r)\lambda(x). \tag{65}$$

In this case the individual vector field σ has the constant direction $\lambda \in \mathscr{H}$ but is of varying length, determined by φ, where φ is allowed to be any smooth functional of the entire forward rate curve. In order to avoid trivialities we make the following assumption.

Assumption 4.32. We assume that $\varphi(r) \neq 0$ for all $r \in \mathscr{H}$.

After a simple calculation the drift vector μ turns out to be

$$\mu(r) = \mathbf{F}r + \varphi^2(r)D - \frac{1}{2}\varphi'(r)[\lambda]\varphi(r)\lambda, \tag{66}$$

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 351)

where $\varphi'(r)[\lambda]$ denotes the Frechet derivative $\varphi'(r)$ acting on the vector λ, and where the constant vector $D \in \mathscr{H}$ is given by

$$D(x) = \lambda(x) \int_0^x \lambda(s)\, ds.$$

We now want to know under what conditions on φ and λ we have a finite dimensional realization, i.e., when the Lie algebra generated by

$$\mu(r) = \mathbf{F}r + \varphi^2(r)D - \frac{1}{2}\varphi'(r)[\lambda]\varphi(r)\lambda, \qquad \sigma(r) = \varphi(r)\lambda,$$

is finite dimensional. Under Assumption 4.32 we can use Lemma 4.28, to see that the Lie algebra is in fact generated by the simpler system of vector fields

$$f_0(r) = \mathbf{F}r + \Phi(r)D, \qquad f_1(r) = \lambda,$$

where we have used the notation

$$\Phi(r) = \varphi^2(r).$$

Since the field f_1 is constant, it has zero Frechet derivative. Thus the first Lie bracket is easily computed as

$$[f_0, f_1](r) = \mathbf{F}\lambda + \Phi'(r)[\lambda]D.$$

The next bracket to compute is $[[f_0, f_1], f_1]$, which is given by

$$[[f_0, f_1], f_1] = \Phi''(r)[\lambda; \lambda]D.$$

Note that $\Phi''(r)[\lambda; \lambda]$ is the second order Frechet derivative of Φ operating on the vector pair $[\lambda; \lambda]$. This pair is to be distinguished from (notice the semicolon) the Lie bracket $[\lambda, \lambda]$ (with a comma), which if course would be equal to zero. We now make a further assumption.

Assumption 4.33. We assume that $\Phi''(r)[\lambda; \lambda] \neq 0$ for all $r \in \mathscr{H}$.

Given this assumption we may again use Lemma 4.28 to see that the Lie algebra is generated by the vector fields

$$f_0(r) = \mathbf{F}r, \qquad f_1(r) = \lambda, \qquad f_3(r) = \mathbf{F}\lambda, \qquad f_4(r) = D.$$

Of these vector fields, all but f_0 are constant, so all brackets are easy. After elementary calculations we see that in fact

$$\{\mu, \sigma\}_{\mathrm{LA}} = \mathrm{span}\{\mathbf{F}r, \mathbf{F}^n\lambda, \mathbf{F}^n D; n = 0, 1, \ldots\}.$$

From this expression it follows immediately that a necessary condition for the Lie algebra to be finite dimensional is that the vector space spanned by $\{\mathbf{F}^n\lambda; n \geq 0\}$ is finite dimensional. This occurs if and only if λ is quasi-exponential. If, on the other hand, λ is quasi-exponential, then we know from Lemma 4.30 that also D is quasi-exponential, since it is the integral of the QE function λ multiplied by the QE function λ. Thus the space $\{\mathbf{F}^n D; n = 0, 1, \ldots\}$ is also finite dimensional, and we have proved the following result.

Proposition 4.34. *Under Assumptions* 4.32 *and* 4.33*, the interest rate model with volatility given by* $\sigma(r, x) = \varphi(r)\lambda(x)$ *has a finite dimensional realization if and only if* λ *is a quasi-exponential function. The scalar field* φ *is allowed to be any smooth field.*

4.5.5. Application 3: when is the short rate a markov process? One of the classical problems concerning the HJM approach to interest rate modeling is that of determining when a given forward rate model is realized by a short rate model, i.e., when the short rate is Markovian. We now briefly indicate how the theory developed above can be used in order to analyze this question. For the full theory see [8].

Using the results above, we immediately have the following general necessary condition.

Proposition 4.35. *The forward rate model generated by* σ *is a generic short rate model, i.e the short rate is generically a Markov process only if*

$$\dim\{\mu, \sigma\}_{\mathrm{LA}} \leq 2. \tag{67}$$

Proof. If the model is really a short rate model, then bond prices are given as $p(t, x) = F(t, R_t, x)$ where F solves the term structure PDE. Thus bond prices, and forward rates are generated by a two dimensional factor model with time t and the short rate R as the state variables. □

Remark 4.36. The most natural case is $\dim\{\mu, \sigma\}_{\mathrm{LA}} = 2$. It has in fact been shown in [27] (see Remark 4.6 therein) that—under some mild and natural technical assumptions—every nontrivial generic short rate model is of dimension 2.

Note that condition (67) is only a necessary condition for the existence of a short rate realization. It guarantees that there exists a two-dimensional realiza-

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 353)

tion, but the question remains whether the realization can chosen in such a way that the short rate and running time are the state variables. This question is completely resolved by the following central result (see [8]).

Theorem 4.37. *Assume that the model is not deterministic, and take as given a time invariant volatility $\sigma(r,x)$. Then there exists a short rate realization if and only if the vector fields $[\mu,\sigma]$ and σ are parallel, i.e., if and only if there exists a scalar field $\alpha(r)$ such that the relation*

$$[\mu,\sigma](r) = \alpha(r)\sigma(r) \tag{68}$$

holds (locally) for all r.

It turns out that the class of generic short rate models is very small indeed. We have, in fact, the following result, which was first proved in [39] (using techniques different from those above). See [8] for a proof based on Theorem 4.37.

Theorem 4.38. *Consider a HJM model with one driving Wiener process and a volatility structure of the form*

$$\sigma(r,x) = g(R,x),$$

where $R = r(0)$ is the short rate. Then the model is a generic short rate model if and only if g has one of the following forms.

- *There exists a constant c such that*

$$g(R,x) \equiv c.$$

- *There exist constants a and c such that*

$$g(R,x) = ce^{-ax}.$$

- *There exist constants a and b, and a function $\alpha(x)$, where α satisfies a certain Riccati equation, such that*

$$g(R,x) = \alpha(x)\sqrt{aR+b}.$$

We immediately recognize these cases as the Ho-Lee model, the Hull–White extended Vasiček model, and the Hull–White extended Cox–Ingersoll–Ross model. Thus, in this sense the only generic short rate models are the affine ones, and the moral of this, perhaps somewhat surprising, result is that most short rate models considered in the literature are not generic but "accidental". To understand the geometric picture one can think of the following program.

(1) Choose an arbitrary short rate model, say of the form

$$dR_t = a(R_t)\,dt + b(R_t)\,dW_t$$

with a fixed initial point R_0.

(2) Solve the associated PDE in order to compute bond prices. This will also produce
- an initial forward rate curve $\hat{r}^o(x)$,
- forward rate volatilities of the form $g(R, x)$.

(3) Forget about the underlying short rate model and take the forward rate volatility structure $g(R, x)$ as given in the forward rate equation.

(4) Initiate the forward rate equation with an arbitrary initial forward rate curve $r^o(x)$.

The question is now whether the thus constructed forward rate model will produce a Markovian short rate process. Obviously, if you choose the initial forward rate curve r^o as $r^o = \hat{r}^o$, then you are back where you started, and everything is fine. If, however, you choose another initial forward rate curve than $\hat{r}^o$, say the observed forward rate curve of today, then it is no longer clear that the short rate will be Markovian. What the theorem above says is that only the models listed above will produce a Markovian short rate model for all initial points in a neighbourhood of $\hat{r}^o$. If you take another model (like, say, the Dothan model) then a generic choice of the initial forward rate curve will produce a short rate process which is no

5. Potentials and positive interest

The purpose of this section is to present two approaches to interest rate theory based on so called "stochastic discount factors" (see below for details), while also relating bond pricing to stochastic potential theory.

An appealing aspect of the approaches described below is that they both generate *positive term structures*, i.e., a system of bond prices for which all induced forward rates are positive.

5.1. Generalities. As a general setup we consider a standard filtered probability space $(\Omega, \mathscr{F}, \boldsymbol{F}, P)$ where P is the objective measure. We now need an assumption about how the market prices various assets.

Assumption 5.1. We assume that the market prices all assets, underlying and derivative, using a fixed martingale measure Q (with the money account as the numeraire).

We now recall that for a T-claim Y the arbitrage free price at $t = 0$ is given by

$$\Pi(0; Y) = E^Q[e^{-\int_0^T r_s\,ds} \cdot Y]. \tag{69}$$

We denote the likelihood process for the transition from the objective measure P to the martingale measure Q by L, i.e.,

$$L_t = \frac{dQ_t}{dP_t},$$

where the index t denotes the restriction of P and Q to $\mathscr{F}_t$. We may of course also write the price in (69) as an expected value under P:

$$E^P[e^{-\int_0^T r_s\,ds} \cdot L_T \cdot Y].$$

This leads us to the following definition.

Definition 5.2. The *stochastic discount factor* (SDF), or *state price density process* Z is defined by

$$Z(t) = e^{-\int_0^t r_s\,ds} \cdot L_t.$$

We now have the following basic pricing result, which follows directly from the Bayes formula.

Proposition 5.3. *For any T-claim Y, the arbitrage free price process is given by*

$$\Pi(t; Y) = \frac{E^P[Z_T Y \mid \mathscr{F}_t]}{Z_t}.$$

In particular, bond prices are given by

$$\Pi(t; Y) = \frac{E^P[Z_T \mid \mathscr{F}_t]}{Z_t}. \tag{70}$$

We now have the following fact which we will use extensively.

Proposition 5.4. *Assume that the short rate is strictly positive and that the economically natural condition $p(0, T) \to 0$ as $T \to \infty$ is satisfied. Then the stochastic discount factor Z is a probabilistic* potential, *i.e.,*

- *Z is a supermartingale.*
- *$E[Z_t] \to 0$ as $t \to \infty$.*

Conversely one can show that any potential will serve as a stochastic discount factor. Thus the moral is that modeling bond prices in a market with positive interest rates is equivalent to modeling a potential, and in the next sections we will describe two ways of doing this.

We end by noticing that we can easily recover the short rate from the dynamics of Z.

Proposition 5.5. *If the dynamics of Z are written as*

$$dZ_t = -h_t\,dt + dM_t,$$

where h is nonnegative and M is a martingale, then the short rate is given by

$$r_t = Z_t^{-1}h_t.$$

Proof. Applying the Itô formula to the definition of Z we obtain

$$dZ_t = -r_t Z_t\,dt + e^{-\int_0^t r_s\,ds}\,dL_t. \qquad \square$$

5.2. The Flesaker–Hughston fractional model. Given a stochastic discount factor Z and a positive short rate we may, for each fixed T, define the process $\{X(t,T); 0 \le t \le T\}$ by

$$X(t,T) = E^P[Z_T \mid \mathcal{F}_t], \tag{71}$$

and thus, according to (70), write bond prices as

$$p(t,T) = \frac{X(t,T)}{X(t,t)}. \tag{72}$$

We now have the following result.

Proposition 5.6. *For each fixed t, the mapping $T \mapsto X(t,T)$ is smooth, and in fact*

$$\frac{\partial}{\partial T}X(t,T) = -E^P[r_T Z_T \mid \mathcal{F}_t]. \tag{73}$$

Furthermore, for each fixed T, the process

$$X_T(t,T) = \frac{\partial}{\partial T}X(t,T)$$

is a negative P-martingale satisfying

$$X_T(0,T) = -p_T(0,T) \quad \textit{for all } T \ge 0.$$

Proof. Using the definition of Z and the Itô formula, we obtain

$$dZ_s = -r_s Z_s\, ds + e^{-\int_0^s r_u\, du}\, dL_s,$$

so

$$Z_T = Z_t - \int_t^T r_s Z_s\, ds + \int_t^T e^{-\int_0^s r_u\, du}\, dL_s.$$

Since L is a martingale, this gives us

$$E^P[Z_T \mid \mathscr{F}_t] = Z_t - E^P\left[\int_t^T r_s Z_s\, ds \mid \mathscr{F}_t\right],$$

and (73) follows immediately. The martingale property now follows directly from (73). □

We can now state the basic result from Flesaker–Hughston.

Theorem 5.7. *Assume that the term structure is positive and*

$$\lim_{T\to\infty} p(t,T) = 0$$

almost surely for all t. Then there exists a family of positive martingales $M(t,T)$ indexed by T and a positive deterministic function Φ such that

$$p(t,T) = \frac{\int_T^\infty \Phi(s) M(t,s)\, ds}{\int_t^\infty \Phi(s) M(t,s)\, ds}. \tag{74}$$

The M family can, up to multiplicative scaling by the Φ process, be chosen as

$$M(t,T) = -X_T(t,T) = E^P[r_T Z_T \mid \mathscr{F}_t].$$

In particular, Φ can be chosen as

$$\Phi(s) = -p_T(0,s), \tag{75}$$

in which case the corresponding M is normalized to $M(0,s) = 1$ for all $s \geq 0$.

Proof. The condition $\lim_{T\to\infty} p(t,T) = 0$ implies that $X(t,T) \to 0$ as $T \to \infty$, so we have

$$X(t,T) = -\int_T^\infty X_T(t,s)\, ds$$

and thus obtain from (72)

$$p(t,T) = \frac{\int_T^\infty X_T(t,s)\,ds}{\int_t^\infty X_T(t,s)\,ds}. \tag{76}$$

If we now define $M(t,T)$ by

$$M(t,T) = -X_T(t,T),$$

then (74) follows from (76) with $\Phi \equiv 1$. The function Φ is only a scale factor which can be chosen arbitrarily, and the choice in (75) is natural in order to normalize the M family. Since X_T is negative, M is positive and we are done. □

There is also a converse of the result above.

Proposition 5.8. *Consider a given family of positive martingales $M(t,T)$ indexed by T and a positive deterministic function Φ. Then the specification*

$$p(t,T) = \frac{\int_T^\infty \Phi(s)M(t,s)\,ds}{\int_t^\infty \Phi(s)M(t,s)\,ds}, \tag{77}$$

defines an arbitrage free positive system of bond prices. Furthermore, the stochastic discount factor Z generating the bond prices is given by

$$Z_t = \int_t^\infty \Phi(s)M(t,s)\,ds.$$

Proof. Using the martingale property of the M family, we obtain

$$E^P[Z_T \mid \mathscr{F}_t] = \int_T^\infty E^P[\Phi(s)M(T,s) \mid \mathscr{F}_t]\,ds = \int_T^\infty \Phi(s)M(t,s)\,ds.$$

This implies, by the positivity of M and Φ, that Z is a potential and can thus serve as a stochastic discount factor. The induced bond prices are thus given by

$$p(t,T) = \frac{E^P[Z_T \mid \mathscr{F}_t]}{Z_t},$$

and the calculation above shows that the induced (arbitrage free) bond prices are given by (77). □

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 359)

The most used instance of a Flesaker–Hughston model is the so called *rational model*. In such a model we consider a given martingale K and two deterministic positive functions $\alpha(t)$ and $\beta(t)$. We then define the M family by

$$M(t,T) = \alpha(T) + \beta(T)K(t).$$

With this specification of M it is easily seen that bond prices will have the form

$$p(t,T) = \frac{A(T) + B(T)K(t)}{A(t) + B(t)K(t)},$$

where

$$A(t) = \int_t^\infty \Phi(s)\alpha(s)\,ds, \qquad B(t) = \int_t^\infty \Phi(s)\beta(s)\,ds,$$

We can specialize this further by assuming K to be of the form

$$K(t) = e^{\int_0^t \gamma(s)\,dW_s - \frac{1}{2}\int_0^t \gamma^2(s)\,ds},$$

where γ is deterministic. Then K will be a lognormal martingale, and the entire term structure will be analytically very tractable.

5.3. Connections to the Riesz decomposition. In Section 5.1 we saw that any stochastic discount factor generating a nice bond market is a potential, so from a modeling point of view it is natural to ask how one can construct potentials from scratch.

The main tool used is the following standard result.

Proposition 5.9 (Riesz decomposition). *If Z is a potential, then it admits a representation as*

$$Z_t = -A_t + M_t, \tag{78}$$

where A is an increasing process, and M is a martingale defined by

$$M_t = E^P[A_\infty \mid \mathscr{F}_t].$$

To construct a potential, let us assume that we define A as

$$A_t = \int_0^t a_s\,ds \tag{79}$$

for some integrable nonnegative process a. Then we easily obtain

$$Z_t = E^P\left[\int_0^\infty a_s\,ds \mid \mathscr{F}_t\right] - \int_0^t a_s\,ds = \int_t^\infty E^P[a_s \mid \mathscr{F}_t]\,ds. \tag{80}$$

We can now connect this to the Flesaker–Hughston framework. The family of processes $X(t,T)$ defined in (71) will, in the present framework, have the form

$$X(t,T) = E^P\left[\int_T^\infty E^P[a_s \mid \mathscr{F}_T]\,ds \mid \mathscr{F}_t\right] = \int_T^\infty E^P[a_s \mid \mathscr{F}_t]\,ds,$$

so the basic family of Flesaker–Hughston martingales are given by

$$M(t,T) = -\frac{\partial}{\partial T}X(t,T) = E^P[a_T \mid \mathscr{F}_t].$$

5.4. Conditional variance potentials. An alternative way of representing potentials which has been studied in depth by Hughston and co-authors is through conditional variances.

Consider a fixed random variable $X_\infty \in L^2(P, \mathscr{F}_\infty)$. We can then define a martingale X by setting

$$X_t = E^P[X_\infty \mid \mathscr{F}_t].$$

Now let us define the process Z by

$$Z_t = E^P[(X_\infty - X_t)^2 \mid \mathscr{F}_t].$$

An easy calculation shows that

$$Z_t = E^P[X_\infty^2 \mid \mathscr{F}_t] - X_t^2.$$

Since the first term is a martingale and the second is a submartingale, the difference is a supermartingale, which by definition is positive and it is in fact a potential.

The point of this is that the potential Z, and thus the complete interest rate model generated by Z, is in fact fully specified by a specification of the single random variable X_∞. A very interesting idea is now to expand X_∞ into Wiener chaos. See the notes in Section 6 below.

5.5. The Rogers Markov potential approach. As we have seen above, in order to generate an arbitrage free bond market model it is enough to construct a positive supermartingale which acts as stochastic discount factor, and in the previous section we saw how to do this using the Riesz decomposition. In this section we will present a systematic way of constructing potentials along the lines above, in terms of Markov processes and their resolvents. The ideas are due to Rogers, and we largely follow his presentation.

We consider a time homogeneous Markov process X under the objective measure P, with infinitesimal generator $\mathscr{G}$.

For any positive real valued sufficiently integrable function g and any positive number α we can now define the process A in the Riesz decomposition (78) as

$$A_t = \int_0^t e^{-\alpha s} g(X_s)\, ds,$$

where the exponential is introduced in order to allow for at least all bounded functions g. In terms of the representation (79) we thus have

$$a_t = e^{-\alpha t} g(X_t),$$

and a potential Z is, according to (80), obtained by

$$Z_t = \int_t^\infty e^{-\alpha s} E^P[g(X_s) \mid \mathscr{F}_t]\, ds.$$

Using the Markov assumption we thus have

$$Z_t = E^P\Big[\int_t^\infty e^{-\alpha s} g(X_s)\, ds \mid X_t\Big], \tag{81}$$

and this expression leads to a well known probabilistic object.

Definition 5.10. For any nonnegative α the *resolvent* R_α is an operator, defined for any bounded measurable function g by the expression

$$R_\alpha g(x) = E_x^P\Big[\int_0^\infty e^{-\alpha s} g(X_s)\, ds\Big],$$

where subscript x refers to the conditioning $X_0 = x$.

We can now connect resolvents to potentials.

Proposition 5.11. *For any bounded nonnegative g, the process*

$$Z_t = e^{-\alpha t} \frac{R_\alpha g(X_t)}{R_\alpha g(X_0)} \tag{82}$$

is a potential with $Z_0 = 1$.

Proof. The normalizing factor is trivial so we disregard it in the rest of the proof. Using time invariance we have, from (81),

$$Z_t = E^P\left[\int_0^\infty e^{-\alpha(t+s)} g(X_{t+s})\, ds \mid X_t\right] = e^{-\alpha t} R_\alpha g(X_t). \qquad \square$$

Given a SDF of the form above, we can of course compute bond prices, and the short rate can easily be recovered.

Proposition 5.12. *If the stochastic discount factor Z is defined by* (82) *then bond prices are given by*

$$p(t,T) = e^{-\alpha(T-t)} \frac{E^P[R_\alpha g(X_T) \mid \mathcal{F}_t]}{R_\alpha g(X_t)}, \tag{83}$$

and the short rate is given by

$$r_t = \frac{g(X_t)}{R_\alpha g(X_t)}. \tag{84}$$

Proof. The formula (83) follows directly from the general formula (70). From the construction of the process a we have

$$dZ_t = -e^{-\alpha t} \frac{g(X_t)}{R_\alpha g(X_0)} dt + dM_t,$$

and (84) now follows from Proposition 5.5. $\square$

One problem with this scheme is that, for a concrete case, it may be very hard to compute the quotient in (84). To overcome this difficulty we recall the following standard result.

Proposition 5.13. *With notation as above we have essentially*

$$R_\alpha = (\alpha - \mathcal{G})^{-1}. \tag{85}$$

The phrase ‘‘essentially’’ indicates that the result is ‘‘morally’’ correct but that care has to be taken concerning the domain of the operators.

Using the identity $R_\alpha = (\alpha - \mathcal{G})^{-1}$ we see that with $f = R_\alpha g$ we have

$$\frac{g(X_t)}{R_\alpha g(X_t)} = \frac{(\alpha - \mathcal{G}) f(X_t)}{f(X_t)},$$

where it usually is a trivial task to compute the last quotient.

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 363)

This led Rogers to use the following scheme.

(1) Fix a Markov process X, number α and a nonnegative function f.

(2) Define g by

$$g = (\alpha - \mathscr{G})f.$$

(3) Choose α (and perhaps the parameters of f) such that g is nonnegative.

(4) Now we have $f = R_\alpha g$, and the short rate can be recaptured by

$$r(t) = \frac{(\alpha - \mathscr{G})f(X_t)}{f(X_t)}.$$

In this way Rogers produces a surprising variety of concrete analytically tractable nonnegative interest rate models, and exchange rate models can also be treated within the same framework.

6. Notes

All basic material in this article can be found in most advanced textbooks, like [1] and [18]. The martingale approach to arbitrage pricing was developed in [31], and [32]. It was then extended in, among other papers, [17], [19]. An elementary textbook on bond markets is [23]. For more advanced treatments see [1] and [18]. The encyclopedic book [11] contains a wealth of theoretical, numerical and practical information. Basic papers on short rate models are [15], [34], [36], [47]. For an example of a two-factor model see [41]. For extensions and notes on the affine term structure theory, see [20]. Jump processes (and affine theory) is treated in [6], [21]. The HJM framework first appeared in [33] and the Musiela parameterization first appeared in [10]. In [6], [4] the HJM theory has been extended to more general driving noise processes. Consistency problems for HJM models and families of forward rate curves were studied in [3], [25], [24], and [26]. The question of when the short rate in a HJM model is in fact Markovian was first studied in [12] for the case of deterministic volatiliy, and for the case of a short rate depending volatility structure it was solved in [39]. The more general question when a given HJM model admits a realisation in terms of a finite dimensional Markovian diffusion was, for various special cases, studied in [44], [13], [37], [5], and [14]. The necessary and sufficient conditions for the existence of finite dimensional Markovian realizations in the general case were first obtained, using methods from differential geometry, in [8], and the problem of actually constructing an FDR is treated in [7]. The functional analytical framework for FDR theory is considerably extended, and many deep and new interesting results are derived, in [27], and [28]. Applications to futures and forward price term structure models are given in [2] and [30].

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 364)

The LIBOR market models were developed in [9], [42]. See also [38] for swap market models. The Flesaker–Hughston models appeared in [29] and analyzed further in [40]. The Wiener chaos approach is developed in [35]. For the Rogers potential approach, see [45].

References

[1] T. Björk, *Arbitrage theory in continuous time*. 2nd ed., Oxford University Press, Oxford 2004. Zbl 1140.91038

[2] T. Björk, M. Blix, and C. Landén, On finite dimensional realizations for the term structure of futures prices. *Internat. J. Theor. Appl. Finance* **9** (2006), 281–314. Zbl 1154.91427 MR 2229823

[3] T. Björk and B. J. Christensen, Interest rate dynamics and consistent forward rate curves. *Math. Finance* **9** (1999), 323–348. Zbl 0980.91030 MR 1849252

[4] T. Björk, G. Di Masi, Y. Kabanov, and W. Runggaldier, Towards a general theory of bond markets. *Finance Stoch.* **1** (1997), 141–174. Zbl 0889.90019

[5] T. Björk and A. Gombani, Minimal realizations of interest rate models. *Finance Stoch.* **3** (1999), 413–432. Zbl 0947.60051 MR 1842281

[6] T. Björk, Y. Kabanov, and W. Runggaldier, Bond market structure in the presence of a marked point processes. *Math. Finance* **7** (1995), 211–239. Zbl 0884.90014 MR 1446647

[7] T. Björk and C. Landén, On the construction of finite dimensional realizations for nonlinear forward rate models. *Finance Stoch.* **6** (2002), 303–331. Zbl 1026.60084 MR 1914314

[8] T. Björk and L. Svensson, On the existence of finite-dimensional realizations for nonlinear forward rate models. *Math. Finance* **11** (2001), 205–243. Zbl 1055.91017 MR 1822777

[9] A. Brace, D. Gątarek, and M. Musiela, The market model of interest rate dynamics. *Math. Finance* **7** (1997), 127–155. Zbl 0884.90008 MR 1446645

[10] A. Brace and M. Musiela, A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton. *Math. Finance* **4** (1994), 259–283. Zbl 0884.90016

[11] D. Brigo and F. Mercurio, *Interest rate models—theory and practice*. Springer Finance, Springer-Verlag, Berlin 2001. Zbl 1038.91040 MR 1846525

[12] A. Carverhill, When is the short rate Markovian? *Math. Finance* **4** (1994), 305–312. Zbl 0884.90017 MR 1299241

[13] O. Cheyette, Markov representation of the Heath-Jarrow-Morton model. Preprint. BARRA Inc., Berkeley 1996. Available at SSRN: http://ssrn.com/abstract=6073

[14] C. Chiarella and O. K. Kwon, Forward rate dependent Markovian transformations of the Heath-Jarrow-Morton term structure model. *Finance Stoch.* **5** (2001), 237–257. Zbl 0972.60058 MR 1841718

[15] J. C. Cox, J. E. Ingersoll, Jr., and S. A. Ross, A theory of the term structure of interest rates. *Econometrica* **53** (1985), 385–407. MR 785475

(AutoPDF V7 26/7/10 09:03) EMS (170×240mm) Tmath J-2324 PMS, 67:3 (idp) PMU:(idp[KN]/A)22/7/2010 pp. 321–367 2324_67-3_02 (p. 365)

[16] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*. Encyclopedia Math. Appl. 44, Cambridge University Press, Cambridge 1992. Zbl 1140.60034 MR 1207136

[17] F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** (1994), 463–520. Zbl 0865.90014 MR 1304434

[18] D. Duffie, *Dynamic asset pricing theory*. 3rd ed., Princeton University Press, Princeton 2001. Zbl 1140.91041

[19] D. Duffie and C.-F. Huang, Multiperiod security markets with differential information: martingales and resolution times. *J. Math. Econom.* **15** (1986), 283–303. Zbl 0608.90006 MR 871158

[20] D. Duffie and R. Kan, A yield-factor model of interest rates. *Math. Finance* **6** (1996), 379–406. Zbl 0915.90014

[21] D. Duffie, J. Pan, and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68** (2000), 1343–1376. Zbl 1055.91524 MR 1793362

[22] M. Elhouar, Finite-dimensional realizations of regime-switching HJM models. *Appl. Math. Finance* **15** (2008), 331–354. Zbl 1142.91520 MR 2451645

[23] F. J. Fabozzi, *Bond markets, analysis, and strategies*. 5th ed., Prentice Hall, Upper Saddle River, N.J., 2004.

[24] D. Filipović, A note on the Nelson-Siegel family. *Math. Finance* **9** (1999), 349–359. Zbl 0997.91034 MR 1849253

[25] D. Filipović, Exponential-polynomial families and the term structure of interest rates. *Bernoulli* **6** (2000), 1081–1107. Zbl 0982.60085 MR 1809736

[26] D. Filipović, *Consistency problems for Heath-Jarrow-Morton interest rate models*. Lecture Notes in Math. 1760, Springer-Verlag, Berlin 2001. Zbl 1008.91038 MR 1828523

[27] D. Filipović and J. Teichmann, Existence of invariant manifolds for stochastic equations in infinite dimension. *J. Funct. Anal.* **197** (2003), 398–432. Zbl 1013.60035 MR 1960419

[28] D. Filipović and J. Teichmann, On the geometry of the term structure of interest rates. *Proc. Roy. Soc. London Ser. A* **460** (2004), 129–167. Zbl 1048.60045 MR 2052259

[29] B. Flesaker and L. Hughston, Positive interest. Risk Magazine **9** (1996), 46–49.

[30] R. M. Gaspar, Finite dimensional Markovian realizations for forward price term structure models. In *Stochastic finance*, Springer, New York 2006, 265–320. Zbl 1153.60041 MR 2230768

[31] J. Harrison and J. Kreps, Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory* **20** (1979), 381–408. Zbl 0431.90019 MR 0540823

[32] J. M. Harrison and S. R. Pliska, Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** (1981), 215–260. Zbl 0482.60097 MR 622165

[33] D. Heath, R. Jarrow, and A. Morton, Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60** (1992), 77–105. Zbl 0751.90009

[34] T. S. Y. Ho and S.-B. Lee, Term structure movements and pricing interest rate contingent claims. *J. Finance* **41** (1986), 1011–1029.

[35] L. P. Hughston and A. Rafailidis, A chaotic approach to interest rate modelling. *Finance Stoch.* **9** (2005), 43–65. Zbl 1063.91033 MR 2210927

[36] J. Hull and A. White, Pricing interest-rate-derivative securities. *Rev. Financ. Stud.* **3** (1990), 573–592.

[37] K. Inui and M. Kijima, A Markovian framework in multi-factor Heath-Jarrow-Morton models. *J. Finan. Quant. Anal.* **33** (1998), 423–440.

[38] F. Jamshidian, Libor and swap market models and measures. *Finance Stoch.* **1** (1997), 293–330. Zbl 0888.60038

[39] A. Jeffrey, Single factor Heath-Jarrow-Morton term structure models based on Markov spot interest rate dynamics. *J. Finan. Quant. Anal.* **30** (1995), 619–642.

[40] Y. Jin and P. Glasserman, Equilibrium positive interest rates: a unified view. *Rev. Financ. Stud.* **14** (2001), 187–214.

[41] F. Longstaff and E. Schwartz, Interest rate volatility and the term structure. *J. Finance* **40** (1992), 1259–1282.

[42] K. Miltersen, K. Sandmann, and D. Sondermann, Closed form solutions for term structure derivatives with log-normal interest rates. *J. Finance* **52** (1997), 409–430.

[43] C. Nelson and A. Siegel, Parsimonious modelling of yield curves. *J. Bus.* **60** (1987), 473–489.

[44] P. Ritchken and L. Sankarasubramanian, Volatility structures of forward rates and the dynamics of the term structure. *Math. Finance* **5** (1995), 55–72. Zbl 0866.90023

[45] L. C. G. Rogers, The potential approach to the term structure of interest rates and foreign exchange rates. *Math. Finance* **7** (1997), 157–176. Zbl 0884.90046 MR 1446646

[46] S. Tappe, Finite dimensional realizations for term structure models driven by semimartingales. PhD Thesis, Humboldt University, Berlin 2005. Zbl 05710617

[47] O. Vasiček, An equilibrium characterization of the term stucture. *J. Finan. Econ.* **5** (1977), 177–188.

[48] F. Warner, *Foundations of differentiable manifolds and Lie groups*. Scott, Foresman & Comp., Glenview, Illinois, London 1971. Zbl 0241.58001 MR 0295244

Received April 15, 2009; revised September 22, 2009

T. Björk, Department of Finance, Stockholm School of Economics, P.O. Box 6501, SE-113 83 Stockholm, Sweden

E-mail: tomas.bjork@hhs.se

R. M. Gaspar, ISEG, Technical University of Lisbon, Rua Miguel Lupi, n. 20, Gab. 101, 1249-078 Lisboa, Portugal

E-mail: Rmgaspar@iseg.utl.pt