MATEMATICAL ECONOMICS - 2013/2014

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Contents

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- Office hours: Thursday 16-18, and Friday 15-17
- Reference text: Shone, Economic Dynamics; Hale and Kocak, Dynamics and Bifurcations

Warning

These notes are in a very preliminary form. Read them notes with some caution, as they are likely to contain mistakes and typos. Corrections are greatly appreciated. Each section of these notes contains exercises. Some extra exercises can be found in the very last section.

1. Introduction

This part of the course concerns differential equations and difference equations. These equations are used to model dynamical processes, e.g., the evolutions of quantities changing in time. If the time is a continuous variable, then the process is modeled by an ordinary differential equation (ODE), whereas if the time is a discrete variable, then the process is modeled by a difference equation (DE).

Example 1.1 (Compound interest). *If an amount A is compounded annually at the interest rate r, then the payment after $t = 1, 2, \dots$ years is given by*

$$P_t = A(1 + r)^t.$$

We see immediately that P_t satisfies the equation:

$$P_{t+1} = (1 + r)P_t.$$

This equation is recursive equation *(or a difference equation), because given P_{t+1} can be computed given P_t.*

Now, if the same amount A is compounded continuously m times each year, then we get

$$P_t = Ae^{rt}.$$

If we think of t as a continuous variable and not just the number of years, then P_t is a solution of the equation

$$\frac{dP_t}{dt} = rP_t,$$

which is a differential equation*, because it involves the derivative dP_t/dt.*

The subject of the differential equations and the difference equation is extensive. These notes focus on a part of the theory of these equations that is called 'qualitative analysis'. The aim is to obtain as much as possible information about an ODE or a DE without looking for explicit solutions.

2. Scalar ODE's

2.1. **Notation.** In the following, the symbol x denotes a real-valued differentiable function $x\colon I \to \mathbb{R}$ on an open interval $I = (a, b)$ of $\mathbb{R}$ with $-\infty \le a < b \le +\infty$, whereas the symbol f denotes a real-valued continuous function $f\colon \mathbb{R} \to \mathbb{R}$. We will use often the notation $\dot{x}(t)$ to denote the derivative dx/dt.

An *ordinary differential equation (ODE)* is an equation relating several quantities: i) a function $t \mapsto x(t)$, ii) some derivatives of $x(t)$, iii) the independent variable t, and iii) other functions of t. The general form of a scalar ODE is the following:

$$\dot{x}(t) = f(t, x(t)) \qquad \text{for } t \in I, \tag{1}$$

where $x\colon I \to \mathbb{R}$ is an unknown function. Equation (5) is called a *scalar ordinary differential equation*. The term 'scalar' means that $x(t)$ is 1-dimensional ($x \in \mathbb{R}$). A function x that satisfies relation (5) is called a *solution* of the differential equation (5).

Most of the time, we will be interested in solutions of (5) such that $x(t_0)$ equals a specific value $x_0 \in \mathbb{R}$ for a specific $t_0 \in \mathbb{R}$. The problem consisting in finding such a solution is called an *initial value problem*,

$$\dot{x} = f(t, x(t)), \qquad x(t_0) = x_0.$$

Example 2.1. *Here are some examples of ODE's:*

(1) $dx/dt = -3x + 4 + e^{-t}$,
(2) $d^2x/dt + 4tdx/dt - 3(1 - t^2)x = 0$,
(3) $dx/dt + 3tx = e^x$.

Some terminology:

- the *order* of a differential equation is the order of the highest derivative appearing in the equation.

- an *nth linear differential equation* is an equation of the form:

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = b(t),$$

  where $a_1, \ldots, a_n$ and g are continuous functions depending only of t. If $a_1, \ldots, a_n$ are constant, then the equation is called a linear differential equation with constant coefficients. If $g \equiv 0$, then the equation is called *homogeneous*, otherwise it is called *non-homogeneous*.
- a differential that is not linear is called *nonlinear*.

Accordingly to the terminology introduced earlier, example (1) is a first order non-homogenous linear differential equation with constant coefficients, example (2) is a second order homogenous linear differential equation, and finally example (3) is a first order nonlinear differential equation.

2.2. **First order linear differential equations.** A first order linear differential equation can be always written in the following form:

$$\dot{x}(t) + a(t)x(t) = b(t). \tag{2}$$

To find the solutions of this equation, we use the method of the integrating factor. By multiplying both sides of the equation by $\alpha(t) := e^{\int a(t)dt}$ (integrating factor), we obtain

$$\alpha(t)\dot{x}(t) + \alpha(t)a(t)x(t) = b(t)b(t).$$

Because of the properties of the exponential, we have $db/dt = \alpha(t)a(t)$. Then, the previous equality can be written as follows,

$$\frac{d}{dt}\left(\alpha(t)x(t)\right) = \alpha(t)b(t).$$

Now, we integrate both sides with respect to t, and obtain

$$\alpha(t)x(t) = \int \alpha(s)b(s)ds + c,$$

where c is the integration constant. The any solution of (2) is given by

$$x(t) = \frac{1}{\alpha(t)}\left(\int \alpha(s)b(s)ds + c\right), \tag{3}$$

for some constant c. The expression (3) is called the *general solution* of (2).

2.3. **First order linear equation with constant coefficients.** Consider a differential equation as in (2) with $a(t) \equiv a$ and $b(t) \equiv b$ for some constants a and b

$$\dot{x} + ax = b. \tag{4}$$

In this case, the integrating factor is $\alpha(t) = e^{\int a dt}$. We can take $\alpha(t) = e^{at}$, and so $\int \alpha(s)b(s)ds = e^{at}b/a$ if $a \neq 0$, and $\int \alpha(s)b(s)ds = bt$ otherwise. From (3), the general solution is then given by

$$x(t) = \begin{cases} \frac{b}{a} + ce^{-at} & \text{if } a \neq 0, \\ bt + c & \text{if } a = 0. \end{cases}$$

If $x(0) = x_0$, then $c = x_0 - b/a$ if $a \neq 0$, and $c = x_0$ otherwise. Hence

$$x(t) = \begin{cases} e^{-at}\left(x_0 - \frac{b}{a}\right) + \frac{b}{a} & \text{if } a \neq 0, \\ bt + x_0 & \text{if } a = 0. \end{cases}$$

is the solution with the initial condition $x(0) = x_0$.

Example 2.2 (Price adjustment demand and supply model)**.** *Consider the following linear model for demand-price and supply-price relations:*

(1) $q_d = A + Bp$ *and* $q_s = C + Dp$ *with* $B < 0$ *and* $D > 0$,

(2) price adjustment equation: $\dot{p} = E(q_d - q_s)$ *with* $E > 0$.

Putting all together, we get

$$\dot{p} = E(B - D)p + E(A - C).$$

Comparing with (4)*, we see that* $a = -E(B-D) > 0$ *and* $b = E(A-C)$*. So*

$$p(t) = e^{-at}\left(x_0 - \frac{b}{a}\right) + \frac{b}{a} \qquad \text{with} \quad \frac{b}{a} = \frac{A - C}{D - B}.$$

It follows that

$$\lim_{t \to +\infty} p(t) = \frac{A - C}{D - B}$$

independently of x_0*.*

2.4. **Scalar autonomous differential equation.** We are interested in equations of the form

$$\dot{x}(t) = f(x(t)) \qquad \text{for } t \in I, \tag{5}$$

where $x\colon I \to \mathbb{R}$ is an unknown function (in particular, the interval I is unknown). Equation (5) is called a *scalar autonomous differential equation.* 'Autonomous' means that f does not depend explicitly on t.

As before, we are interested in the solutions of the initial value problem:

$$\dot{x} = f(x), \qquad x(t_0) = x_0. \tag{6}$$

The equation (5) has the following mechanical interpretation: if $x(t)$ denotes the position of a point-particle on the real line $\mathbb{R}$, then $\dot{x}(t)$ is the instantaneous speed of the particle. Thus the first part of (6) says that the value of the speed of the particle at time t is equal to $f(x(t))$, and so it depends on its position. The second part of (6) says that the position of the particle at time t_0 is equal to x_0.

Remark 2.3. *Note that if $x(t)$ is a solution of (6), then if we define $\bar{x}(t) = x(t + t_0)$, then $\bar{x}$ is a solution of (6) with $\bar{x}(0) = x(t_0) = x_0$ (Check it). For this reason, we could always take $t_0 = 0$ in (6). From the geometrical point of view, the transformation $x(t) \mapsto x(t+t_0)$ corresponds to translate the graph of x along the t-axis to the left by a length t_0.*

2.5. **Separation of variables.** To solve problem (6), we can argue as follows.

We consider separately two cases: 1) $f(x_0) = 0$, and 2) $f(x_0) \neq 0$.

Case 1: According to our mechanical model, $f(x_0) = 0$ mens that the velocity of the particle has to be zero when the particle is at x_0. But this implies that the particle cannot moves away and changes its position. Therefore out mechanical model suggests that the function $x(t) = x_0$ for every $t \in \mathbb{R}$ has to be the wanted solution. To check that this is correct is easy. In fact, $\dot{x}(t) = 0$ and $x(0) = x_0$.

Case 2: Since f is continuous, we have $f(x) \neq 0$ around x_0, and so as long as t is closed to 0, we can divide both sides of (6) by $f(x(t))$. Hence,

$$\frac{\dot{x}(t)}{f(x(t))} = 1.$$

We then integrate both sides of the previous equation from 0 to t (with t not too far from 0),

$$\int_0^t \frac{\dot{x}(s)}{f(x(s))} ds = t,$$

and by substitution $u = x(s)$, we obtain

$$\int_{x_0}^{x(t)} \frac{du}{f(u)} = t. \tag{7}$$

Since $f(x_0) \neq 0$ and f is continuous, then $f(u) > 0$ for $u \in (x_0, x(t))$ or $f(u) < 0$ for $u \in (x_0, x(t))$ (we are assuming that $x(t) > x_0$, but the argument remains the same if $x(t) < x_0$). It follows that the function

$$F(z) := \int_{x_0}^{z} \frac{du}{f(u)}$$

is strict monotone. But $F(x(t))$ is precisely the left-hand side of (7). So if F^{-1} is the inverse of F, then we see that the solution $x(t)$ of (7) (for t close to 0) is given by $x(t) = F^{-1}(t)$ (verify that this is indeed a solution of (6)).

Exercise 2.4. *Solve the following initial value problem using the method of separation of variables. Plot the solutions. Is the solution unique in*

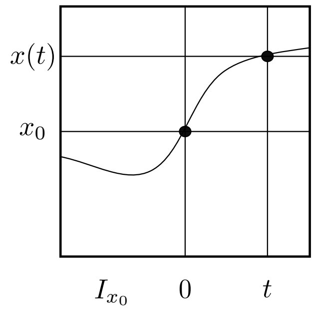


FIGURE 1. Solution $x(t, x_0)$ on the interval I_{x_0}

exercise (c)?

$$\begin{array}{lll} a) & \dot{x} = -x & x(0) = 1. \\ b) & \dot{x} = x^2 & x(0) = -1. \\ c) & \dot{x} = \sqrt{x} & x(0) = 0. \\ d) & \dot{x} = x(1-x) & x(0) = 1. \end{array}$$

2.6. **Existence and uniqueness of solutions.**

Definition 2.5. *The symbol C^0 denoted the sets of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, and the symbol C^1 denotes the subset of C^0 of all differentiable functions with continuous derivatives $f\colon \mathbb{R} \to \mathbb{R}$.*

As explained in Remark 2.3, there is no loss of generality in assuming that $t_0 = 0$. So, unless specified otherwise, we take $t_0 = 0$ from now on.

Theorem 2.6.
(1) *Suppose that $f \in C^0$. Then for every x_0, there exist an interval (possibly infinite) $I_{x_0} = (a_{x_0}, b_{x_0})$ containing $t_0 = 0$ and a solution $x\colon I_{x_0} \to \mathbb{R}$ of the initial value problem* (6).
(2) *Suppose that $f \in C^1$. Then in addition to (1), the solution x is unique and differentiable with continuous derivative.*

The largest possible interval I_{x_0} is called the *maximal interval of existence of the solution* (see Fig. 1). We will use the notation $x(t, x_0)$ to denote the solution of (6) with $x(0) = x_0$.

2.7. **Phase portrait.**

Definition 2.7. *Let $x_0 \in \mathbb{R}$, and let $x(t, x_0)$ be the solution with initial condition x_0. The set $\gamma(x_0) = \bigcup_{t \in (a_{x_0}, b_{x_0})} x(t, x_0)$ is called the* orbit of x_0. *The collection of the orbits of all points $x_0 \in \mathbb{R}$ is called the* phase portrait *of* (5).

Definition 2.8. *A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point of* (5) *if $f(\bar{x}) = 0$.*

Suppose that $x(t, x_0) = x_0$ for every $t \in \mathbb{R}$. Then the orbit $\gamma(x_0)$ consists of the single point $\{x_0\}$. If $x(t, x_0) \equiv x_0$, then x_0 has to be an equilibrium point.

Exercise 2.9. *Determine the phase portrait of the following differential equations. Note that $f \in C^1$ in each example so that the existence and uniqueness of the initial value problem is guaranteed by Theorem 2.6.*

(1) $\dot{x} = x$,
(2) $\dot{x} = x - x^3$,
(3) $\dot{x} = 1 + x$,
(4) $\dot{x} = x(1 - x)$,
(5) $\dot{x} = -x + x^3 + \lambda$ *with* $\lambda \in \mathbb{R}$,
(6) $\dot{x} = 1 - \sin x$.

2.8. **Equilibrium points and their stability.**

Definition 2.10. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (5) *is* stable *if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - \bar{x}| < \delta$, then the solution $x(t, x_0)$ of* (5) *satisfies $|x(t, x_0) - \bar{x}| < \epsilon$ for every $t \geq 0$.*

Definition 2.11. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (5) *is* asymptotically stable *if it is stable, and there exists $r > 0$ such that if $|x_0 - \bar{x}| < r$, then* $\lim_{t\to+\infty} x(t, x_0) = \bar{x}$.

Definition 2.12. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (5) *is called* unstable *if it is not stable.*

The following theorem is a stability criterion for equilibria in terms of the derivative of f.

Theorem 2.13. *Suppose that $f \in C^1$ and $\bar{x} \in \mathbb{R}$ is an equilibrium point of* (5).

(1) If $f'(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.
(2) If $f'(\bar{x}) > 0$, then $\bar{x}$ is unstable.

An equilibrium point $\bar{x}$ is called *hyperbolic* if $f'(\bar{x}) \neq 0$, and non-hyperbolic if $f'(\bar{x}) = 0$.

Remark 2.14. *Note that Theorem 2.13 does not say anything when the equilibrium point $\bar{x}$ is not hyperbolic. In this case, one should look at higher order derivatives of f at $\bar{x}$. For example, try to determine the stability of the equilibrium point of $\dot{x} = x^3$.*

Exercise 2.15. *Determine the type (hyperbolic or non-hyperbolic) and the stability of the equilibria in Exercises 2.9.*

The next lemma summarizes the main properties of the solution $x(t, x_0)$, and can be proved by using phase portrait analysis.

Lemma 2.16. *The solution $x(t, x_0)$ has the following properties:*

(1) $x(t, x_0)$ is monotone in t,

(2) $x(t, x_0)$ *is increasing in* x_0*, i.e,* $x(t, x_0) < x(t, y_0)$ *if* $x_0 < y_0$*,*

(3) if $x(t, x_0)$ *is bounded for every* $t \geq 0$ $(t \leq 0)$*, then* $b_{x_0} = +\infty$ $(a_{x_0} = -\infty)$ *and* $\lim_{t\to+\infty} x(t, x_0) = \bar{x}$ $(\lim_{t\to-\infty} x(t, x_0) = \bar{x})$ *with* $\bar{x}$ *being an equilibrium point (i.e.,* $f(\bar{x}) = 0$*).*

2.9. **Linear ODE's.** Let $a, b \in \mathbb{R}$. Consider the linear differential equation

$$\dot{x} = ax + b. \tag{8}$$

The equation is called *linear homogeneous* if $b = 0$, and *linear non-homogeneous* if $b \neq 0$.

To obtain the solution $x(t, x_0)$ of (8) satisfying the initial condition $x(0) = x_0$, one may argue as follows. If $a = 0$, then (8) becomes $\dot{x} = b$, and by integrating both side of this equation from 0 to t, we immediately obtain

$$x(t, x_0) = x_0 + bt.$$

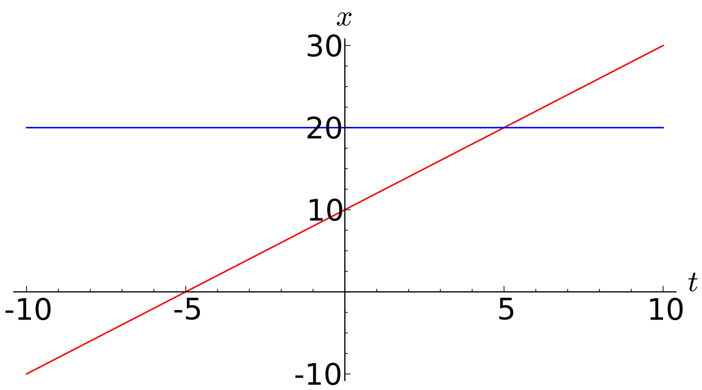
Now, suppose that $a \neq 0$. Since $f(x) = ax + b$, the equation has a unique one equilibrium point $\bar{x} = -b/a$. If we define $y(t, y_0) = x(t, x_0) - \bar{x}$, then $\dot{y} = \dot{x} = ax + b = a(x - \bar{x}) = ay$. The solution $\dot{y} = ay$ satisfying $y(0) = y_0$ is equal to $y(t, y_0) = y_0 e^{at}$. But $y_0 = x_0 - \bar{x}$, and so we can conclude that $x(t, x_0) = y(t, y_0) + \bar{x} = (x_0 - \bar{x})e^{at} + \bar{x}$, i.e.,

$$x(t, x_0) = \left(x_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}. \tag{9}$$

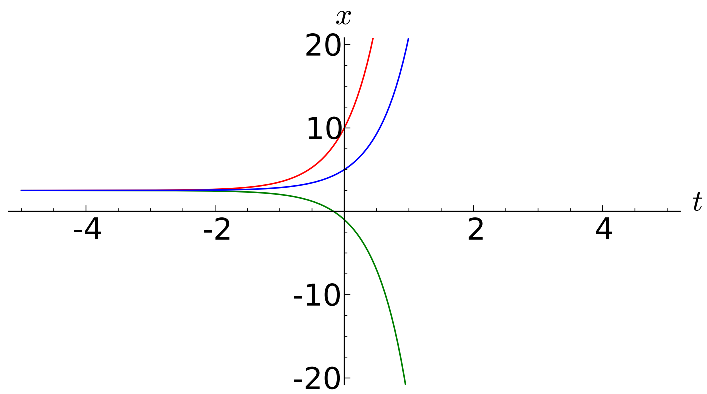
(Find the same solution using the method of separation of variables explained in Subsection 2.5)

Examples.

(1) Suppose that $a = 0$. The next figure depicts the solutions $x(t, 10)$ for $b = 2$ (red) and the solution $x(t, 20)$ for $b = 0$ (blue).



(2) Suppose that $a = 2$ and $b = 1$. Then $\bar{x} = -1/2$ is the (unique) equilibrium point, and the solutions $x(t, 1)$, $x(t, -1)$ and $x(t, -1/2)$ computed using (9) are depicted in the figure below.



2.10. **Additional exercises.**

(1) Assume that a population $p(t)$ grows at a constant rate k. This means that $p(t)$ satisfies the following differential equation:

$$\dot{p}(t) = kp(t).$$

Find the solution $p(t, p_0)$, determine the phase portrait of the equation and the stability of its equilibrium point.

(2) According to a continuos version of the Harrod-Domar economy growth model, the relation between the savings S, the income Y and the investment I is given by

$$S = sY, \quad I = \nu\dot{Y}, \quad I = S,$$

where s and ν are constants denoting the average propensity to save and the coefficient of the investment relationship, respectively. Derive and solve the differential equation for $Y(t)$. Determine its phase portrait and the stability of its equilibrium point.

(3) Prove the claim in Remark 2.3.

(4) Prove Lemma 2.16.

3. Scalar DE's

Difference equations (DE's) are the analog of differential equations when the time is a discrete variable $n = 0, 1, \ldots$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1 real-valued function.

Example 3.1. *The following is an example of a difference equation arising from a financial problem.*

Let p_n be the price of some financial assets at time $n = 0, 1, 2, \ldots$. Suppose that the variation of p_n in time is given by the following arbitrage condition*:*

$$(1 + r)p_n = d + p^e_{n+1}, \tag{10}$$

where $r > 0$ is the rate of return, $d > 0$ is the dividend, and p^e_{n+1} is the expected price at time $n + 1$. Suppose also that the agents have *perfect foresight, i.e., they know that the mechanism of price formation is given by the following relation*

$$p^e_{n+1} = p_{n+1}. \tag{11}$$

We want to determine how p_n *varies in time.*

By combining (10) *and* (11)*, we obtain a difference equation for* p_n *only:*

$$(1+r)p_n = d + p_{n+1}.$$

This equation can be written as

$$p_{n+1} = F(p_n), \qquad \text{where } F(p) = (1+r)p + d.$$

This is the DE describing the evolution of p_n*.*

3.1. **General form.** A scalar DE is a recursive equation of the form:

$$x_{n+1} = F_n(x_n) \qquad \text{for } n = 0, 1, \ldots, \tag{12}$$

where F_n is a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. If $F_n = F$ for every n, then we say that the DE is *autonomous*. Otherwise, we say that DE is not autonomous. Equation (12) is called a difference equation, because it can be written in such a way that its right-hand side can be rewritten as a difference:

$$x_{n+1} - x_n = F_n(x_n) - x_n.$$

3.2. **Autonomous DE's.** A DE is called *autonomous* if $F_n = F$ for every $n = 0, 1, \ldots$. In this case, (12) becomes

$$x_{n+1} = F(x_n) \qquad \text{for } n = 0, 1, \ldots. \tag{13}$$

If F is continuous, then we can define

$$F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}},$$

and the solution of (13) with the initial condition $x_0 = z$ is

$$x_n = F^n(z).$$

When such a solution exists, it is clear that it is also unique.

Definition 3.2. *The union of all elements* $x_0, F(x_0), F^2(x_0), \ldots$ *is called the* positive orbit of x_0*, and is denoted by* $\gamma^+(x_0)$*.*

Remark 3.3. *Although the initial value problem* $x_{n+1} = F(x_n)$ *with* $x_0 = z$ *has a unique solution, it may be possible for two solutions to coincide from some time on. This is not possible for ODE's (why?).*

3.3. **Linear DE's.** A linear difference equation is an equation of the form:

$$x_{n+1} = ax_n + b, \qquad a, b \in \mathbb{R}. \tag{14}$$

The equation is called *homogeneous* if $b = 0$, and *non-homogeneous* if $b \neq 0$.

Solutions of these equations can be computed explicitly. Note first that Equation (14) has a (unique) fixed point $\bar{x} = b/(1-a)$ if and only if $a \neq 1$. The orbit of (14) is given by

$$x_n = \begin{cases} x_0 + nb & \text{if } a = 1, \\ a^n(x_0 - \bar{x}) + \bar{x} & \text{otherwise.} \end{cases} \tag{15}$$

This includes the case $a = 0$ for which the orbit consists of the fixed point $\bar{x} = b$.

3.4. **Terminal value problem.** If $F\colon \mathbb{R} \to \mathbb{R}$ is a bijection, then $F^{-1}\colon \mathbb{R} \to \mathbb{R}$ is well defined, and we can consider another DE generated by the map F^{-1}:

$$x_n = F^{-1}(x_{n+1}) \qquad \text{for } n = 0, 1, \ldots. \tag{16}$$

In this case, we can consider the *terminal value problem*, which consists in solving (16) with the terminal value condition $x_m = z$ for some $m \in \mathbb{N}$.

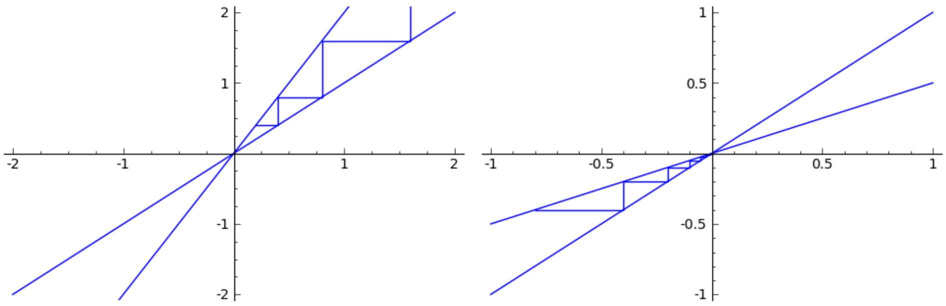
3.5. **Examples.**

(1) linear $F(x) = ax + b$ with $a, b \in \mathbb{R}$,
(2) quadratic (logistic): $F(x) = ax(1-x)$ with $a > 0$,
(3) power systems: $F(x) = cx^a$ with $c > 0$ and $a > 0$,
(4) piecewise linear system: $F(x) = 1 - 2|x - 1/2|$.

3.6. **Stair-step diagram.** The stair-step diagram is a geometrical method for depicting the orbits of a DE. The method is illustrated in the following examples.

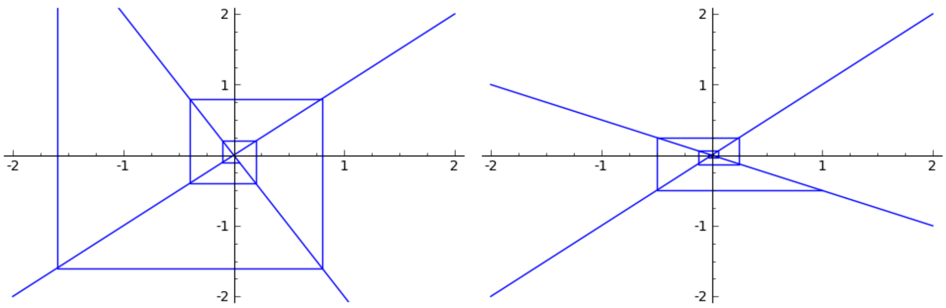
3.7. **Fixed points and oscillating behavior.** We saw that the asymptotic behavior of solutions of an autonomous scalar ODE can be understood by studying the stability properties of the equilibrium points. For DE's, the analog role is played by fixed points. These points can help understand the asymptotic behavior of some orbits but not all of them. It is worth pointing out that autonomous scalar maps exhibit a more complicated dynamics then ODE's. For instance, they may have orbits with oscillating behaviors, like periodic orbits (see examples (2) and (4) below).

Example 3.4. *(1)* $x_{n+1} = 2x_n$*. The orbits of this map can be computed explicitly. By iterating* F*, we obtain* $x_n = 2^n x_0$ *for* $x_0 \in \mathbb{R}$*. The step-stair diagram for this map is depicted in Fig. 2(A).*



(A) (B)

FIGURE 2. (A) $x_{n+1} = 2x_n$ with $x_0 = 0.2$. (B) $x_{n+1} = x_n/2$ with $x_0 = -0.8$.



(A) (B)

FIGURE 3. (A) $x_{n+1} = -2x_n$ with $x_0 = 0.01$. (B) $x_{n+1} = -x_n/2$ with $x_0 = 1$.

(2) $x_{n+1} = x_n/2$. *The orbits of this maps are* $x_n = 2^{-n}x_0$ *for* $x_0 \in \mathbb{R}$ *(see Fig. 2(B))*

(3) $x_{n+1} = -2x_n$. *The orbits of this maps are* $x_n = (-2)^n x_0$ *for* $x_0 \in \mathbb{R}$ *(see Fig. 3(A)). Compare these orbits with those of the previous examples. Note the oscillatory behavior of the orbits in this example and the next.*

(4) $x_{n+1} = -x_n/2$. *The orbits of this maps are* $x_n = (-2)^{-n}x_0$ *for* $x_0 \in \mathbb{R}$ *(see Fig. 3(B)).*

Definition 3.5. *A point* $\bar{x}$ *is called a fixed point of* F *if* $F(\bar{x}) = \bar{x}$.

Definition 3.6. *A point* $\bar{x} \in \mathbb{R}$ *is called a* periodic point *of* (14) *of period* $m > 0$ *if* $F^m(\bar{x}) = \bar{x}$, *i.e., if* $\bar{x}$ *is a fixed point of the map* F^m.

Remark 3.7. *Note that* $\bar{x}$ *is a fixed point of* F *if and only if* $\gamma^+(\bar{x}) = \bar{x}$.

3.8. **Stability.** As for equilibrium points of differential equations, we can define the notions of a stability, instability and asymptotic stability for periodic points. We will focus on fixed points. Since every periodic point is a fixed point of a certain iterate of F, it is easy how to extend the definitions and results presented below to periodic points.

Definition 3.8. *Let $\bar{x} \in \mathbb{R}$ be a fixed of the map F. Then $\bar{x}$ is called*

(1) stable *if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - \bar{x}| < \delta$, then the orbit x_n satisfies $|x_n - \bar{x}| < \epsilon$ for every $n \geq 0$;*
(2) asymptotically stable *if it is stable and there exists $r > 0$ such that if $|x_0 - \bar{x}| < r$, then the orbit x_n satisfies* $\lim_{t \to +\infty} x_n = \bar{x}$*;*
(3) if it is not stable.

The following theorem is a stability criterion in terms of the derivative of F.

Theorem 3.9. *Suppose that $F \in C^1$ and $\bar{x} \in \mathbb{R}$ is a fixed point of F.*

(1) If $|F'(\bar{x})| < 1$, then $\bar{x}$ is asymptotically stable.
(2) If $|F'(\bar{x})| > 1$, then $\bar{x}$ is unstable.

A fixed point $\bar{x}$ is called *hyperbolic* if $|F'(\bar{x})| \neq 1$, and *non-hyperbolic* if $|F'(\bar{x})| = 1$.

Remark 3.10. *To determine the stability of a non-hyperbolic fixed point $\bar{x}$, we need to compute derivatives of F of order ≥ 2. But we do not get a simple criterion as Theorem 3.9.*

3.9. **Exercises.**

(1) Some of the fixed points are non-hyperbolic, and therefore Theorem 3.9 cannot be used. Use instead the stair-step diagram.
   (a) Find the fixed points of $F(x) = x + x^2$ and determine their stability.
   (b) Find the fixed points of $F(x) = -x + 3x^2$ and determine their stability. Hint: consider $F^2(x)(= x - 18x^3 + 27x^4)$, the second iterate of F.
   (c) Derive Formula (15). This can be done using a method similar to that one used to obtain the solutions of linear differential equations in Subsection 2.9.
(2) Suppose that $F\colon I \to I$ is a bijection (surjective and invertible), and consider the DE $x_{n+1} = F(x_n)$ for $n \geq 0$. Let m be a positive integer, and let x. Find the initial condition $x_0 \in \mathbb{R}$ such that $x_m = x$.
(3) Pick $2 < a < 3$, and consider the difference equation with $F(x) = ax(1 - x)$. Find the fixed points, and study their stability. Are there periodic points of period 2? What can you say about the asymptotic behavior of the remaining orbits?
(4) Consider piecewise linear system $F(x) = 1 - 2|x - 1/2|$. Find fixed points and periodic points of periodic 2. Then study their stability. Hint: for the periodic point of period 2, compute first F^2. To do that, consider compute F^2 on the intervals $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$.

(5) Let F be as the previous exercise, and suppose that $\bar{x}$ is periodic point of F, i.e., $F^m(\bar{x}) = \bar{x}$. Then what can you say about the stability of $\bar{x}$? Hint: use the derivative criterion and the product rule for derivatives.

(6) Use a computer to plot some orbits of the logistic for several $1 < a < 4$. Then try $a = 4$. Try to describe the behavior of the orbits. Do they converge to fixed points or periodic orbits? (if you are not a computer wizard, the computation can be performed directly by wolframalpha. Google wolframalpha and type logistic map. The parameter a is called r there. Pick different r's and the initial condition x_0's and see what happens.)

4. Planar ODE's

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function, and let $x\colon I \to \mathbb{R}^2$ be a differentiable function on the interval $I = (a, b)$ with $-\infty \le a < b \le +\infty$. We are interested in the solutions of the autonomous differential equation:

$$\dot{x}(t) = f(x(t)), \qquad t \in I. \tag{17}$$

4.1. **Homogeneous linear ODE's with constant coefficients.**

More specifically, we are interested in the case $f(x) = Ax$ with A being a 2×2 matrix with constant real coefficients, and x being a vector of $\mathbb{R}^2$:

$$\dot{x} = Ax. \tag{18}$$

If we write

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then Equation (18) takes the form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2. \end{aligned}$$

4.2. **General properties of linear systems.**

(1) Existence and uniqueness: the solution $x(t, x_0)$ of Equation (18) with initial condition $x(0) = x_0 \in \mathbb{R}^2$ exists and it is unique. Its maximal interval of existence is the entire real line $\mathbb{R}$.

(2) Superposition Principle: if x and y are two solutions of (18), then every linear combination $c_1x + c_2y$ with $c_1, c_2 \in \mathbb{R}$ is a solution as well. This is simple to prove. Let $z = c_1x + c_2y$. Then $\dot{z} = c_1\dot{x} + c_2\dot{y}$. Since x and y are solutions of (18), we have $\dot{z} = c_1Ax + c_2Ay$. But $c_1Ax + c_2Ay = A(c_1x + c_2y) = Az$, and we can conclude that $\dot{z} = Az$, i.e., z is a solution.

(3) In analogy to the scalar case, the solution of (18) with initial condition $x_0 \in \mathbb{R}^2$ is given by

$$x(t, x_0) = e^{tA}x_0 \qquad \forall t \in \mathbb{R},$$

where e^{tA} is a matrix (for its definition, see the next subsection).

4.3. **Exponential of a matrix.** It is a fact that the series $\sum_{n=0}^{+\infty} A^n/n!$ converges for every 2×2 matrix A. This allows us to define the exponential of a matrix as follows.

Definition 4.1. *For every 2×2 matrix A, we define*

$$e^A = \sum_{n=0}^{+\infty} \frac{A^n}{n!}.$$

Of course if A is a matrix and t is a real number, then tA is still a matrix. The main properties of the matrix e^{tA} are the following:

(1) $e^{(s+t)A} = e^{sA}e^{tA}$ for $s, t \in \mathbb{R}$,
(2) $de^{tA}/dt = Ae^{tA} = e^{tA}A$,
(3) if $AB = BA$ (i.e., A and B commute), then $e^{t(A+B)} = e^{tA}e^{tB}$.

Exercise 4.2. *Show that if $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$, then $e^{t(A+B)}$ and $e^{tA}e^{tB}$ do not coincide.*

4.4. **Exponential of Normal Jordan Forms.**

Definition 4.3. *Every matrix having one of the following three forms is called a* Jordan Normal Form*,*

$$(i) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad (iii) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda, \alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

We now compute e^{tA} when A is a Normal Jordan Form.

Form (i): It follows directly from the definition of e^{tA} that

$$e^{tA} = \begin{pmatrix} \sum_{n=0}^{+\infty} \dfrac{(t\lambda_1)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{+\infty} \dfrac{(t\lambda_2)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}.$$

Form (ii): We can write $A = I + \lambda N$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since I and N commute, it follows from Property (3) of e^{tA} that

$$e^{tA} = e^{\lambda I} e^{tN} = e^{\lambda t} e^{tN}.$$

Now, we see that $N^2 = 0$ (i.e., N^2 is the matrix with zero entries). This implies that $N^k = 0$ for $k \geq 2$, and so

$$e^{tN} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$e^{tA} = e^{t\lambda} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Form (iii): We can write $A = \alpha I + \beta K$, where $K = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since I and K commute, Property (3) of e^{tA} implies that

$$e^{tA} = e^{\alpha t} e^{\beta t K}.$$

Now, check that $K^2 = -I$ and $K^3 = -K$. From this, we get $K^{2n} = (-1)^n I$ and $K^{2n+1} = (-1)^n K$, and so

$$\begin{aligned} e^{tK} &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\beta t)^{2n} K^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\beta t)^{2n+1} K^{2n+1} \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\beta t)^{2n} \right) I + \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\beta t)^{2n+1} \right) K \\ &= \cos(\beta t) I + \sin(\beta t) K = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}. \end{aligned}$$

Finally,

$$e^{tA} = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

4.5. **Phase portrait.** We now draw the phase portrait of the differential equation $\dot{x} = Ax$ when A is one of the Normal Jordan Forms introduced in Subsection 4.4. Although the phase portrait is the collection of all the orbits of the equation, we do not need to plot all the of them, but only a few representative ones. Since we know that the general solution of the equation is $x(t, x_0) = e^{tA} x_0$ with $x(0) = x_0$, all that we need to do is to understand the geometry of the transformation of the plane $x_0 \mapsto e^{tA} x_0$, sending the vector x_0 into the new vector $e^{tA} x_0$.

Form (i): It is quite easy to understand the geometrical effect of the transformation e^{tA} in this case. Its effect is that of multiplying the first component of the vector x by $e^{t\lambda_1}$ and the second component of x by $e^{t\lambda_2}$. Depending on the sign of λ_1 and λ_2, the phase portrait is depicted in Fig. 4 (cases: saddle, sink and source).

Form (ii): The transformation e^{tA} can be thought as the compositions of two transformations: $e^{t\lambda} x$ and $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) x$. The first transformation expands or contracts x depending on the sign of λ, whereas the second transformation 'slides' the vector $x = (x_1, x_2)$ along the horizontal line $y = x_2$. The overall effect of e^{tA} produces the phase portrait (improper node) depicted in Fig. 5.

Form (iii): The geometry of $e^{tA} x$ is the combination of the expansion or contraction generated by $e^{\alpha t}$ with the rotation of the plane generated by the matrix $\left(\begin{smallmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{smallmatrix}\right)$ (clockwise if $\beta > 0$ and counterclockwise if $\beta < 0$). The phase portrait is depicted in Fig. 4 (cases: spiral sink, spiral source and center).

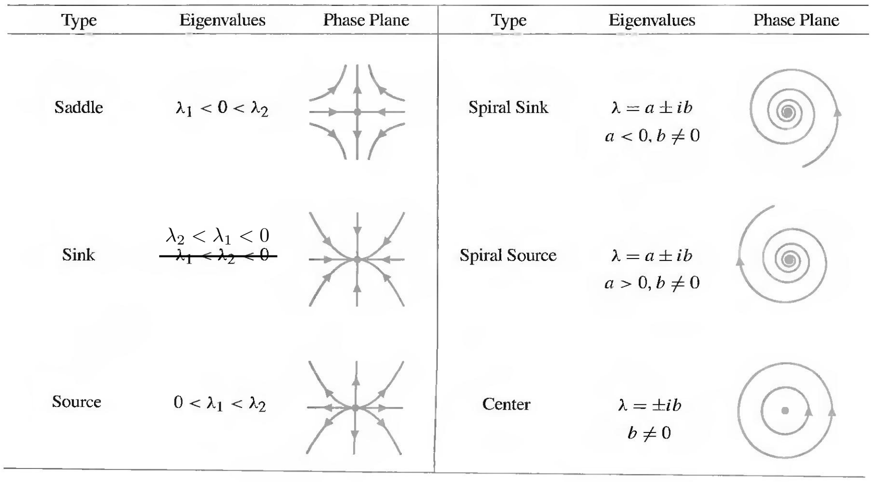


FIGURE 4. Phase Portraits.

4.6. **Change of coordinates.** Suppose that x is a solution of the differential equation $\dot{x} = Ax$. Let P be an invertible real 2×2 matrix, and define $y = P^{-1}x$. Then, y is a solution of the differential equation:

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy.$$

The general solution of this equation is $y(t) = e^{tP^{-1}AP}y_0$ for $y_0 \in \mathbb{R}^2$. This implies that $x(t) = Pe^{tP^{-1}AP}P^{-1}x_0$, where $x_0 = Py_0 = x(0)$. But we know that solution $x(t, x_0)$ is given by $x(t, x_0) = e^{tA}x_0$, and so we conclude that

$$e^{tA} = Pe^{tP^{-1}AP}P^{-1}.$$

Now, suppose that given a real matrix A, we can find an invertible matrix P such that $P^{-1}AP$ is a Jordan Normal form. So in order

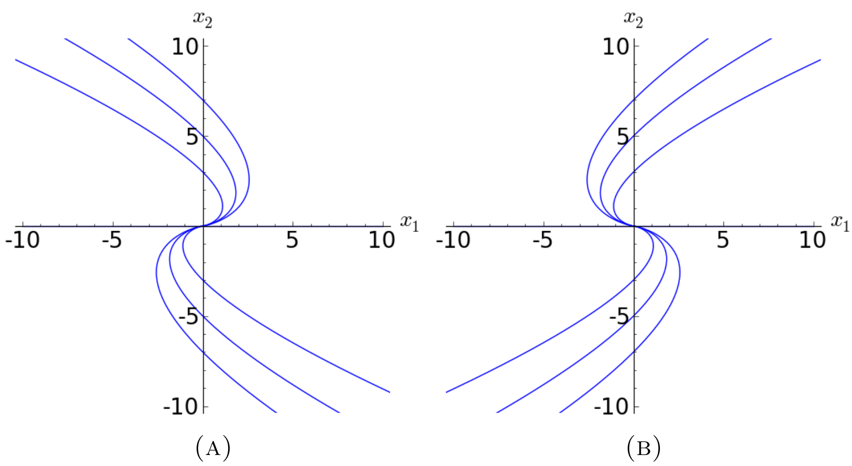


FIGURE 5. Improper Node. (A) $\lambda < 0$. (B) $\lambda > 0$.

to compute e^{tA}, we do not have to compute directly e^{tA}, but we can simply compute $Pe^{tP^{-1}AP}P^{-1}$, and we know that from Subsection 4.4 that $e^{tP^{-1}AP}$ is one of the matrices:

$$\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}, \qquad e^{t\lambda}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad e^{\alpha t}\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

Exercise 4.4. *Consider the linear differential equation*

$$\dot{x} = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix},$$

and the change of coordinates $y = P^{-1}x$ with $P = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Find the differential equation in the new coordinates y, compute the general solution for this equation, and finally derive the general solution in the coordinates x.

4.7. Jordan Decomposition Theorem.

Theorem 4.5. *Suppose that A is a real 2×2 matrix. There exists and invertible real 2×2 matrix P such that $P^{-1}AP = J$, and J is one of the following matrices:*

$$(i) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad (iii) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with $\lambda_1, \lambda_2, \lambda, \alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. The matrix J is called a Normal Jordan form.

We now explain how to compute the matrix P. The procedure consists of three steps:

Step 1: Find the eigenvalues of A, which are solutions of the characteristic equation:

$$\det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0, \tag{19}$$

where $\operatorname{tr}(A)$ and $\det(A)$ are the trace and determinant of A, respectively. This is a quadratic equation with real coefficients, and so it has two solutions λ_1 and λ_2 that can be of one of the following types:

- **(a):** λ_1, λ_2 real and $\lambda_1 \neq \lambda_2$,
- **(b):** $\lambda_1 = \lambda_2 = \lambda$ real,
- **(c):** $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with α, β real and $\beta \neq 0$, i.e., λ_1 and λ_2 are complex conjugate.

Step 2: Find the eigenvectors of A. This can be done for each case (a), (b) and (c) as follows.

- **(a):** Since $\lambda_1 \neq \lambda_2$, the matrix A is diagonalizable. This means that A has two linearly independent eigenvectors v_1 and v_2 corresponding to the eigenvalues λ_1 and λ_2, respectively. These vectors are non-zero solutions of the equations:

$$(A - \lambda_i I)\, v_i = 0, \qquad i = 1, 2.$$

(b): We have two subcases. The first corresponds to the situation when A admits two linearly independent eigenvectors v_1 and v_2, that is, when two linearly independent vectors v_1 and v_2 are solutions of the equation

$$(A - \lambda I)\, v = 0. \tag{20}$$

The second subcase corresponds to the situation when any two non-zero solutions of equation (20) are linearly dependent. In this case, let v_1 be a non-zero solution of (20), and let v_2 be any non-zero vector such that

$$(A - \lambda I)\, v_2 = v_1. \tag{21}$$

The vector v_1 is an eigenvector of A, and v_2 is called a *generalized eigenvector of* A.

(c): Let v be an eigenvector of A corresponding to the eigenvalue $\alpha + i\beta$. It turns out that the components of v are complex numbers. So we can write $v = v_1 + iv_2$, where v_1 and v_2 are vectors with real components.

Step 3: Let v_1 and v_2 be the vectors computed for each case in Step 2. Then $P = (v_1|v_2)$. This means that v_1 and v_2 are the first column and the second column of P, respectively. From the construction of v_1 and v_2 in Step 2, these vectors are linearly independent (can you explain why?), and so P is invertible. The Jordan Normal form J associated to A is given by $J = P^{-1}AP$.

4.8. **Stability criterion for linear ODE's.**

Theorem 4.6. *Let A be a real 2×2 matrix. Then the origin $(0, 0)$ is always an equilibrium point of the equation $\dot{x} = Ax$. Furthermore,*

(1) if all the eigenvalues of A have negative real parts, then the origin is asymptotically stable;
(2) if at least one of the eigenvalues of A has positive real part, then the origin is unstable.

Exercise 4.7. *Consider the linear differential equation $\dot{x} = Ax$. For each of the cases below, find the matrix P and the Jordan Normal form J for A. Then sketch the phase portrait of the equation in the new coordinates $y = P^{-1}x$, and determine the stability of the equilibrium point $(0, 0)$. Finally, compute $e^{tA} = Pe^{tJ}P^{-1}$. How many equilibrium points does the equation have in exercise iv)?*

i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *ii)* $\frac{1}{2}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, *iii)* $\begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$, *iv)* $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

4.9. **Non-homogeneous linear differential equations.** Let A be a real 2×2 matrix, and let $b\colon \mathbb{R} \to \mathbb{R}^2$ be a continuous functions. Also, le x_0 be a vector of $\mathbb{R}^2$. Consider the initial value problem for the non-homogeneous linear equation:

$$\dot{x}(t) = Ax(t) + b(t), \qquad x(0) = x_0 \tag{22}$$

To find the solution of this problem, we use the method of *variation of constants*. We look for a solution of the form $x(t) = e^{tA}Z(t)$, where $Z(t)$ is a vector of $\mathbb{R}^2$ depending on $t \in \mathbb{R}$. By replacing such a solution in (22), the two sides of that equation become $\dot{x} = Ae^{tA}Z + e^{tA}\dot{Z}$, and $Ax + b = Ae^{tA}Z + b$. By equating and multiplying both sides by $e^{-tA} = (e^{tA})^{-1}$, we obtain

$$\dot{z} = e^{-tA}b.$$

The vector z can be now computed by integrating between 0 and t. We obtain

$$z(t) = z(0) + \int_0^t e^{sA}b(s)ds.$$

Since $z(0) = x_0$. The solution $x(t)$ is given by

$$x(t) = e^{tA}\left(x_0 + \int_0^t e^{-sA}b(s)ds\right).$$

Exercise 4.8. *(1) Solve the initial value problem* (22) *for*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(Solution: $x_1(t) = 2te^t$ and $x_2(t) = -1 + 2e^t$, where $x(t) = (x_1(t), x_2(t))$.)

(2) Suppose that the matrix A is invertible real 2×2 matrix, and that b does not depend on t. Then (22) *has a unique equilibrium point given by $\bar{x} = -A^{-1}b$ (check this). Show that the solution of* (22) *can be written as $x(t) = \bar{x} + e^{tA}(x_0 - \bar{x})$.*

4.10. **Second order scalar linear ODE's.** A general non-homogeneous second order scalar linear differential equation with constant coefficients is an equation of the form

$$\ddot{x} + a\dot{x} + bx = g(t), \tag{23}$$

where a and b are real constants, and $g(t)$ is a continuous function of t. Note that $x(t)$ is just a real number here, and not a vector of $\mathbb{R}^2$. The initial conditions for a second order differential equations are $x(0) = x_0$ and $\dot{x}(0) = x_0'$.

The general solution of this equation can be found by reducing it to a planar non-homogeneous first order linear equation with constant coefficients. This is how it can be done. Let $y = \dot{x}$. Then we have

$\dot{y} = \ddot{x} = -a\dot{x} - b\dot{x} = g(t)$. Now, if we define the 2×2 matrix A, and the vectors $X(t), h(t)$ of $\mathbb{R}^2$ to be

$$A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \qquad X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad h(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix},$$

then we easily see that

$$\dot{X} = AX + h(t) \qquad \text{and} \qquad X(0) = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix},$$

which is a first order planar linear ODE satisfying the initial condition $X(0) = (x_0, x_0')$. The solution $X(t)$ of this equation is given in Subsection 4.9. The first component $x(t)$ of $X(t)$ is the solution of (23) with initial condition $x(0) = x_0$ and $\dot{x}(0) = x_0'$.

Exercise 4.9. *Find the general solution of the following second order differential equations:*

(1) $\ddot{x} + bx = 0$ with $b > 0$ (harmonic oscillator without friction),
(2) $\ddot{x} + a\dot{x} + bx = 0$ with $a, b > 0$ (harmonic oscillator with friction).

5. Planar DE's

Let $F_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a sequence of continuous transformation, and let x_n be a vector of $\mathbb{R}^2$. Then a *planar difference equation* is the recursive equation given by

$$x_{n+1} = F_n(x_n) \qquad \text{for } n = 0, 1, 2, \ldots. \tag{24}$$

As in the scalar case, we say that the equation is autonomous if there exists a transformation f such that $F_n = F$ for every $n = 0, 1, 2, \ldots$.

5.1. Linear DE's.

Let $F(x) = Ax + b$ with A and b being a constant 2×2 matrix and a constant vector of $\mathbb{R}^2$, respectively. By iterating (24), one can easily see that its solution (with initial condition x_0) is given by

$$x_n = A^n x_0 + (I + A + \cdots + A^{n-1})b \qquad \text{for } n \geq 1.$$

If $\det(I - A) \neq 0$, then $(I - A)^{-1}$ exists, and we have

$$\begin{aligned} I + A + \cdots + A^{n-1} &= (I + A + \cdots + A^{n-1})(I - A)(I - A)^{-1} \\ &= (I - A^n)(I - A). \end{aligned}$$

The solution of (24) can then be written as follows:

$$x_n = A^n x_0 + (I - A^n)(I - A)^{-1}b.$$

5.2. **Computation of A^n.** Given a real 2×2 matrix A, the Jordan Decomposition Theorem (Theorem 4.5) guarantees the existence of a canonical Jordan form J and a real invertible matrix P such that $A = PJP^{-1}$. Then

$$A^n = PJP^{-1}PJP^{-1} \cdots PJP^{-1} = PJ^nJ^{-1}. \tag{25}$$

Moreover, the matrix J takes one of the following forms:

$$(i) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad (iii) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}$$

with $\lambda_1, \lambda_2, \lambda, \alpha, \beta$ real numbers and $\beta \neq 0$. We now compute J^n.

Case (i): we immediately obtain

$$J^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}.$$

Case (ii): if we write $J = I + N$ with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then the Binomial formula gives $J^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} N^k = \lambda^n + n\lambda^{n-1}N$ because $N^2 = 0$. Hence,

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

Case (iii): let $\rho = \sqrt{\alpha^2 + \beta^2}$. Then we can write

$$J = \rho \begin{pmatrix} \alpha/\rho & \beta/\rho \\ -\beta/\rho & \alpha/\rho \end{pmatrix}.$$

But $(\alpha/\rho)^2 + (\beta/\rho)^2 = 1$, and so there exists $\theta \in [0, 2\pi)$ such that $\alpha/\rho = \cos\theta$ and $\beta/\rho = \sin\theta$. Hence,

$$J = \rho \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Now $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ is the matrix of a clockwise rotation of an angle θ, and the nth power of such a matrix is again a rotation of angle $n\theta$. We conclude that

$$J^n = \rho^n \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

5.3. **Phase portrait of homogeneous linear DE's.** We explain how to derive the phase portrait of homogeneous ($b \equiv 0$) linear DE's when A is equal to one of the normal Jordan forms (i)-(iii) above. The general case can the be derived by understanding the geometrical action of the change of coordinates P.

We will only discuss the a few cases, from which though, one should be able to deduce the phase portrait for the general case. Namely, we suppose that $\lambda_1, \lambda_2, \lambda > 0$. We explain below that J^n can be written as $e^{tJ'}$ for some matrix J' and some t. Having written J^n as an exponential of a matrix, the phase portrait of J^n can be obtained from that of the exponential of Jordan canonical forms discussed in Subsection 4.5.

(1) Case (i): we can write $J^n = e^{tJ'}$ with $J' = \left(\begin{smallmatrix} \cos\log\lambda_1 & 0 \\ 0 & \cos\log\lambda_2 \end{smallmatrix}\right)$ and $t = n$.
(2) Case (ii): check that $J^n = e^{tJ'}$ with $J' = \left(\begin{smallmatrix} \lambda\log\lambda & 1 \\ 0 & \lambda\log\lambda \end{smallmatrix}\right)$ and $t = n/\lambda$.
(3) Case (iii): it is easy to see that $J^n = e^{tJ'}$ with $J' = \left(\begin{smallmatrix} \log\rho & \theta \\ -\theta & \log\rho \end{smallmatrix}\right)$ and $t = n$.

5.4. **Stability criterion for linear DE's.** The stability criterion for the homogeneous linear DE

$$x_{n+1} = Ax_n,$$

is as for the corresponding homogeneous linear ODE (Theorem 4.6)

$$\dot{x} = Ax,$$

but with λ_1 and λ_2 replaced by $\log|\operatorname{Re}\lambda_1|$ and $\log|\operatorname{Re}\lambda_2|$.

Exercise 5.1. *Consider the homogenous linear DE's*

$$x_{n+1} = Ax_n$$

with A equal to

$$(i)\quad \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}, \qquad (ii)\quad \begin{pmatrix} 4 & -2 \\ 1 & -1 \end{pmatrix}, \qquad (iii)\quad \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

For each case,

- *find the fixed points of A,*
- *determine their stability,*
- *compute x_n with $x_0 = (1, 0)$.*

5.5. **Second order scalar linear DE's.** The approach taken here to finding a solution of a second order linear DE's is very similar to the one used to finding solutions of second order linear ODE's in Subsection 4.10. We

Exercise 5.2. *The Fibonacci sequence consists of the following numbers $1, 1, 2, 3, 5, 8, 13, \ldots$. Such a sequence can be generated as follows*

$$x_n = x_n + x_{n-1} \qquad \text{for } n \geq 2$$

and with $x_0 = 0$ and $x_1 = 1$. This is a second order scalar DE. Using the method described in Subsection 4.10 but applied to the Fibonacci DE, compute x_n, and show that $\lim_{n\to+\infty} x_{n+1}/x_n = (1 + \sqrt{5})/2$.

6. Extra exercises

6.1. **Scalar ODE's.**

(1) Solve the initial value problem using the method of separation of variables.

$$
\begin{aligned}
(i) \quad & \dot{x} = \frac{1}{x^2}, && x(0) \neq 0,\\
(ii) \quad & \dot{x} = x(x-2), && x(0) = x_0,\\
(iii) \quad & \dot{x} = \frac{1}{2\sqrt{x}}, && x(0) = x_0 \geq 0.
\end{aligned}
$$

(2) For each of the following differential equations find all the equilibrium points and determine whether they are stable, asymptotically stable or unstable. Also, draw the phase portrait.

$$
\begin{aligned}
(i) \quad & \dot{x} = x^3 - 3x,\\
(ii) \quad & \dot{x} = x^4 - x^2,\\
(iii) \quad & \dot{x} = \cos x,\\
(iv) \quad & \dot{x} = \sin^2 x,\\
(v) \quad & \dot{x} = |1 - x^2|.
\end{aligned}
$$

(3) The following differential equations depends on a parameter a. Plot the phase portrait for $a = -1$, $a = 0$ and $a = 1$.

$$
\begin{aligned}
(i) \quad & \dot{x} = x^2 - ax,\\
(ii) \quad & \dot{x} = x^3 - ax.
\end{aligned}
$$

(4) Solve the following linear non-homogeneous equations

$$
\begin{aligned}
(i) \quad & \dot{x} = 2x + 3, && x(0) = 10,\\
(ii) \quad & \dot{x} = -x + 2, && x(0) = -10,\\
(iii) \quad & \dot{x} = 3x + 10, && x(0) = 2.
\end{aligned}
$$

6.2. **Scalar maps.**

(1) For each of the following difference equations, draw the stair-step diagram and plot some iterations. Establish whether the fixed point is stable, asymptotically stable or unstable. Explain why. In which of these examples does the system oscillate around the fixed point?

$$
\begin{aligned}
(i) \quad & 10 - 3x_n = 2 + x_{n-1},\\
(ii) \quad & 25 - x_{n+1} = 3 + 4x_{n-1},\\
(iii) \quad & 45 - 2.5x_{n+1} = 5 + 7.5x_{n-1}.
\end{aligned}
$$

(2) For the following difference equations, draw the stair-step diagram, and iterates 4 times the initial condition $x_0 = .4$. Determine whether the fixed points are stable, asymptotically stable

or unstable.

$$(i) \quad x_{n+1} = 4x_n(1 - x_n),$$
$$(ii) \quad x_{n+1} = x_n^2 - 2,$$
$$(iii) \quad x_{n+1} = -2\left|x - \frac{1}{2}\right| + 1.$$

6.3. **Planar ODE's.**

(1) Sketch the phase portrait of the equation $\dot{x} = Ax$ for the following matrices. Determine the stability of the origin, and compute the exponential matrix e^{tA}.

$$a) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad b) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \qquad c) \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$
$$d) \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad e) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad f) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(2) For each of the following linear equations $\dot{x} = Ax$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find the matrix P such that $J = P^{-1}AP$ is a Jordan Normal form.
   (c) Compute the exponential matrices e^{tJ} and e^{tA}.
   (d) Find the solution $x(t, x_0)$ with initial condition x_0.
   (e) Sketch the phase portrait for the system $\dot{y} = Jy$.
   (f) Determine the stability of the origin $(0, 0)$.

$$a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c) \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$
$$d) \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \qquad e) \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}, \qquad f) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(3) Solve the initial value problem: $\dot{x}_1 = -4x_2$, $\dot{x}_2 = x_1$ with $x_1(0) = 0$ and $x_2(0) = -7$.

(4) Find all the solutions of the linear non-homogeneous system: $\dot{x}_1 = x_2$, $\dot{x}_2 = 2 - x_1$.

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