Nonlinear equations

Definition

A nonlinear equation is any equation of the form

$$f(x) = 0$$

where f is a nonlinear function.

Nonlinear equations

- $x^2 + x + 1 = 0$ $(f : \mathbb{R} \to \mathbb{R})$
- $(x - \cos y, 2y - \sin x) = (0, 0)$ $(f : \mathbb{R}^2 \to \mathbb{R}^2)$
- $y'(t) = \cos(y(t))$ $(f : C^1(\Omega) \to C^0(\Omega), \quad f(y) = y' - \cos(y))$

Remark

A nonlinear equation can have any number of solutions (finite, countable, uncountable)

Example

Suppose that in the beginning of each year a bank client makes a deposit of v euros in an investment fund (constant interest rate) and at the end of n years withdraws M euros. What was the interest rate r? Since we have

$$M = v\frac{1+r}{r}[(1+r)^n - 1],$$

the answer to our question is the solution of the equation

$$f(r) = 0, \qquad \text{where } f(r) = M - v\frac{1+r}{r}[(1+r)^n - 1]$$

Example (Implied volatility)

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

$$\begin{aligned} N(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left[\ln(S/K) + (r + \sigma^2/2)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

$C(S,t)$: option price, r: interest rate free of risk, $T-t$: time to maturity, S: value of the underlying asset, K: strike.
The **implied volatility** is computed considering that every other parameter is known.

The fixed point method in $\mathbb{R}$

The fixed point method applies to the solution of equations in the form

$$g(x) = x$$

Any point $z \in D_g$ that satisfies the equation is called a fixed point of g. The name is due to the fact that the application of g does not change the value of z.

The fixed point method

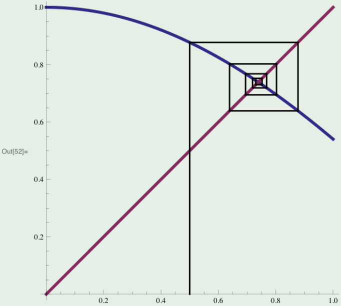
1. Pick an initial guess x_0.
2. Apply function g getting a sequence defined recursively by $x_{n+1} = g(x_n)$.
3. The solution to the equation is $z = \lim x_n$.

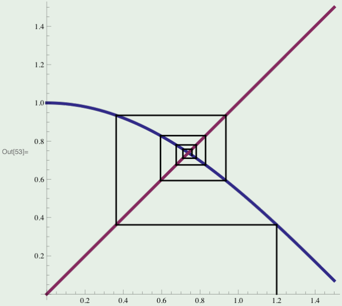
Remark

Does the limit exist? Is it really the solution of the equation?

Example

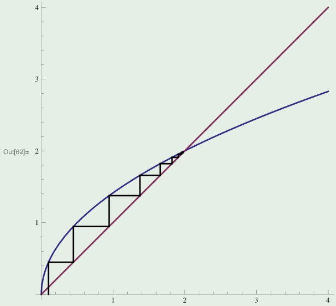
$$\cos x = x$$

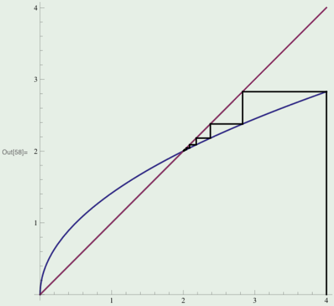




Example

$$\sqrt{2x} = x$$





But does this always work ??

Example

$$-\frac{x^2}{4}-\frac{x}{2}+4=x, \qquad x_0=1.97$$

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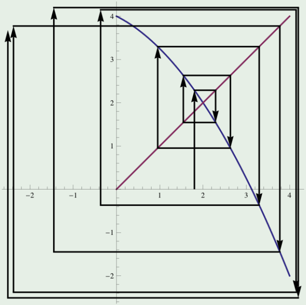


Theorem (Fixed point theorem)

Let $g : [a, b] \to \mathbb{R}$ be a continuous function. Then

1. *If $g([a, b]) \subseteq [a, b]$ then g has at least one fixed point in $[a, b]$.*
2. *Additionally, if g is contractive * in $[a, b]$, it has one and only one fixed point in $[a, b]$, given by the limit of the sequence $x_{n+1} = g(x_n)$, for all $x_0 \in [a, b]$.*

*Contractivity

A function $g : [a, b] \to \mathbb{R}$ is contractive if there is a constant $0 \geq L < 1$ such that

$$|g(x) - g(y)| \leq L|x - y|, \qquad \forall x, y \in [a, b]$$

If g is differentiable this condition is equivalente to existing a constant $0 \leq L < 1$ such that $|g'(x)| < 1, \quad \forall x \in [a, b]$.

Error bounds

Theorem (FPT: Error estimates)

Under the assumptions of the fixed point theorem, we have

1. $|z - x_n| \leq L|z - x_{n-1}|, \quad n \geq 1$
2. $|z - x_n| \leq L^n|z - x_0|, \quad n \geq 1$
3. $|z - x_n| \leq \frac{L}{1-L}|x_n - x_{n-1}|, \quad n \geq 1$
4. $|z - x_n| \leq \frac{L^n}{1-L}|x_1 - x_0|, \quad n \geq 1$

Estimates (1), (2) are called a priori estimates whereas estimates (3),(4) are called a posteriori estimates.

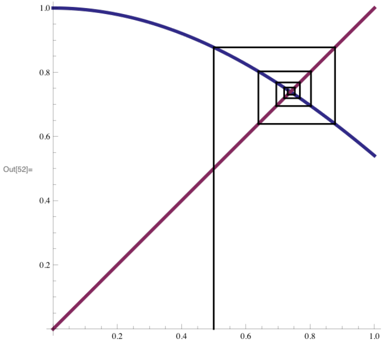
Remark

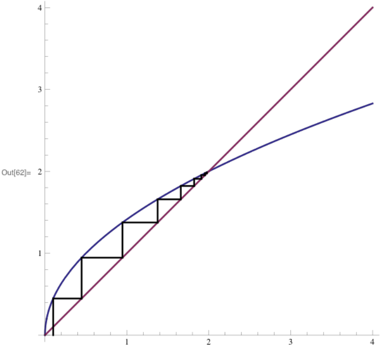
- *If* $|g'(z)| < 1$ *there is a neighbourhood of* z *where the assumptions of the FPT hold.*
- *If* $|g'(z)| > 1$ *the FPM diverges, unless* $x_k = z$*, for some* k*.*

Monotonic convergence

Under the same conditions of the fixed point method:

- If $-1 < g'(x) < 0, \forall x \in [a, b]$ the convergence is alternate and the fixed point is always between consecutive iterations.
- If $0 < g'(x) < 1$ the convergence is monotonous.





Rate of convergence

A sequence $x_n \to z$ is said to converge with rate p if

$$\lim \frac{|z - x_n|}{|z - x_{n-1}|^p} = K(\neq 0)$$

The constant K is denoted as the asymptotic convergence factor.

Theorem

Let g be a function with p continuous derivatives verifying the FPT assumptions and z its fixed point. If

$$g'(z) = \cdots g^{(p-1)}(z) = 0, g^{(p)}(z) \neq 0$$

then the FPM has rate of convergence p.

Using Taylor's polynomial we can check that if a fixed point method has order p then

$$\lim \frac{|z - x_n|}{|z - x_{n-1}|^p} = \lim_{n\to\infty} \frac{g^{(p+1)}(\eta_n)}{(n+1)!} = \frac{g^{(p)}(z)}{(p+1)!}$$

which means that

$$|z - x_n| \approx \frac{|g^{(p+1)}(z)|}{(p+1)!}|z - x_{n-1}|^p$$

Remark

If $p = 1$ *we have* $|e_n| \leq C|e_{n-1}|$.
If $p = 2$ *we have* $|e_n| \leq C|e_{n-1}|^2$.

Newton's method

Newton's method is a particular case of the FPM to solve equations of the type $f(x) = 0$. The method is based on the observation that if $f'(x) \neq 0$ then

$$f(x) = 0 \Leftrightarrow -\frac{f(x)}{f'(x)} = 0 \Leftrightarrow x = x - \frac{f(x)}{f'(x)}$$

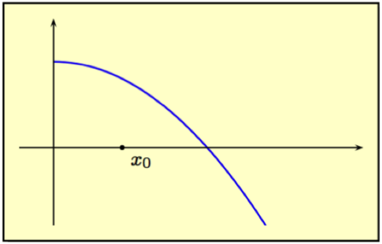
Remark

If $f' \neq 0$, every zero of $f(x)$ is a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$

Newton's Method

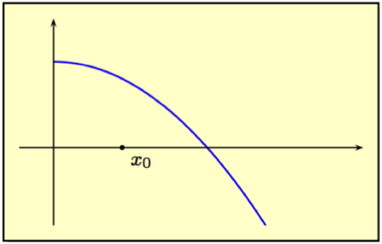
1. Choose an initial approximation x_0
2. For $n \geq 0$ compute $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
3. Repeat until convergence (up to a prescribed accuracy).

Geometric interpretation

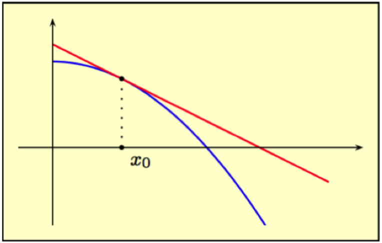


Choose the initial approximation

Geometric interpretation



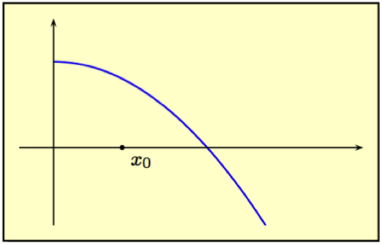
Choose the initial approximation



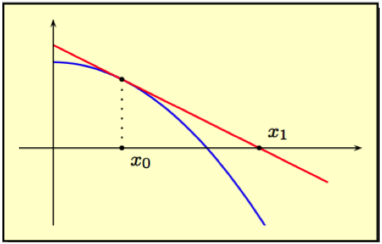
Take the tangent line to the graphic at $x = x_0$. The equation of the tangent is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Geometric interpretation



Choose the initial approximation



Take as new approximation the point of the tangent line crosses de x-axis

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Convergence of Newton's method

Local convergence

If $f \in C^2(V_\varepsilon(z))$ and $f'(z) \neq 0$ Newton's method converges to z at least with order 2. More precisely,

$$|z - x_{n+1}| = \frac{f''(\eta)}{2f'(x_n)}|z - x_n|^2 \leq K|z - x_n|^2,$$

where $K = \dfrac{\max|f''|}{2\min|f'|}$. Applying this estimate recursively, we get

$$|z - x_n| \leq \frac{1}{K}(K|z - x_0|)^{2^n}.$$

Global convergence

Suppose that $f : [a,b] \to \mathbb{R}$ is $C^2([a,b])$ and

1. $f(a) \cdot f(b) < 0$
2. $f'(x) \neq 0, x \in [a,b]$
3. $f''(x) \geq 0$ or $f''(x) \leq 0, \quad x \in [a,b]$

Then,

1. Choosing $x_0 \in [a,b]$ such that $f(x_0)f''(x) \geq 0$, Newton's method is convergent.
2. If $\frac{|f(a)|}{|f'(a)|} < b-a$ and $\frac{|f(b)|}{|f'(b)|} < b-a$ the method converges for every initial approximation $x_0 \in [a,b]$.

Systems of equations

Determine $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ satisfying

$$F(x) = 0 \Leftrightarrow \begin{cases} f_1(x_1, \cdots, x_n) = 0 \\ f_2(x_1, \cdots, x_n) = 0 \\ \cdots \\ f_n(x_1, \cdots, x_n) = 0 \end{cases}$$

or

$$x = G(x) \Leftrightarrow \begin{cases} x_1 = g_1(x_1, x_2, \cdots, x_n) \\ x_2 = g_2(x_1, x_2, \cdots, x_n) \\ \cdots \\ x_n = g_n(x_1, x_2, \cdots, x_n) \end{cases}$$

Theorem (Fixed point in $\mathbb{R}^n$)

Let $\Omega \subset \mathbb{R}$ a nonempty closed set and $G : \Omega \to \Omega$ a contractive function for some norm $\| \cdot \|$. Then, the system $x = G(x)$ has one and only one solution $z \in \Omega$, the fixed point iteration $x_{x+1} = G(x_n)$ converges to z, for every $x_0 \in \Omega$ and the error bounds are similar to the ones obtained in $\mathbb{R}$.

Remark

A function $G : \Omega \to \Omega$ is contractive in a norm $| \cdot \|$ if there exists a constant $0 \leq L < 1$ such that $\|G(x) - G(y)\| \leq L\|x - y\|$

Remark

If G is differentiable, $L = \sup_{x \in \Omega} \|J_G(x)\|$.

Definition (Norm)

Let X be a real vector space. An application $\|\cdot\| : X \to \mathbb{R}_0^+$ is called a **norm** if:

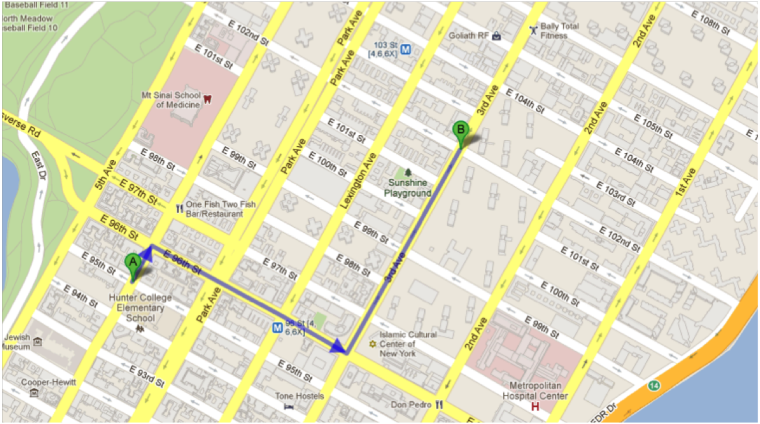
- $\|x\| = 0$ if and only if $x = 0$.
- $\|\alpha x\| = |\alpha|\|x\|, \qquad \forall x \in X, \forall \alpha \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|, \qquad \forall x, y \in X$.

Example (Norms in $\mathbb{R}^n$.)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$\|x\|_\infty = \max_{i=1,\cdots,n} |x_i|$$

Why should we need different norms?



$$d(A, B) = \|A - B\| = \cdots$$

Matrix Norms

Norms in spaces of linear operators

If U and V are normed spaces then $\mathcal{L}(U,V)$ is a normed space considering the norm

$$\|A\| = \sup_{x\in X\backslash 0} \frac{\|Ax\|_Y}{\|x\|_X}, \qquad \forall A \in \mathcal{L}(X,Y)$$

If $U = V = \mathbb{R}^n$, the linear operators are represented by matrices

Matrix norms

$$\|A\|_1 = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|X\|_1} = \max_{j=1,\cdots,n} \sum_{i=1}^{n} |A_{ij}|$$

$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{i=1,\cdots,n} \sum_{j=1}^{n} |A_{ij}|$$

Matrix norms

Proposition

Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ any matrix norm. Then we have $\|A\| \geq |\lambda|$.

Proposition

If all the eigenvalues λ_i of $A \in \mathbb{R}^{n\times n}$ satisfy $|\lambda_i| < 1$ then, for some norm, we have $\|A\| < 1$.

Remark

If $G : \Omega \subset \mathbb{R}^n \to \Omega$ has enough regularity and all the eigenvalues $\lambda_i(x)$ of $J_G(x)$ satisfy $|\lambda_i(x)| < 1, \quad x \in \Omega$, the system $G(x) = x$ has one and only one solution in Ω and the fixed point method converges to this solution.

Example

A pharmaceutical company synthesises an antiviral serum based on the active substances S_1, S_2, S_3 and S_4. It has been determined that if an individual is given α milligrams of the serum, after a fixed amount of time the concentration of each substance (mg/l) is given implicitly (for $\alpha \in [0, 5]$) by the relations

$$\begin{cases} 16x_1 - \cos\left(\alpha\left(x_2 - 2x_1\right)\right) = 0 \\ 16x_2 + 0.75\sin\left(\alpha\left(-x_3 - 3x_1\right)\right) = 0 \\ 16x_3 - \cos\left(\alpha\left(x_4 - 2x_3\right)\right) = 0 \\ 16x_4 - 0.75\sin\left(2\alpha x_3\right) = 0 \end{cases}$$

Knowing that S_2 should never exceed a concentration of $0.03mg/l$ what is the highest safe dosaje that can be administered?

The system can be written as

$$\begin{cases} x_1 = \frac{1}{16}\cos\left(\alpha\left(x_2 - 2x_1\right)\right) \\ x_2 = \frac{3}{64}\sin\left(\alpha\left(x_3 + 3x_1\right)\right) \\ x_3 = \frac{1}{16}\cos\left(\alpha\left(x_4 - 2x_3\right)\right) \\ x_4 = \frac{3}{64}\sin\left(2\alpha x_3\right) \end{cases}$$

Observing the system we can immediately see that, if there is a solution $z \in \mathbb{R}^4$, then

$$z \in [-\frac{1}{16}, \frac{1}{16}] \times [-\frac{3}{64}, \frac{3}{64}] \times [-\frac{1}{16}, \frac{1}{16}] \times [-\frac{3}{64}, \frac{3}{64}]$$

In fact, we can be more precise... For instance,

$$(x_1, x_2) \in [-\frac{1}{16}, \frac{1}{16}] \times [-\frac{3}{64}, \frac{3}{64}] \Rightarrow \alpha(x_2 - 2x_1) \in [\frac{-55}{64}, \frac{55}{64}]$$
$$\Rightarrow \cos\alpha(x_2 - 2x_1) \in [0.6; 1] \Rightarrow x_1 \in [0.0375; 0.0625]$$

Making similar calculations for the other variables we can choose to search a solution in the compact set

$$\Omega = [0.0375; 0.0625] \times [0.044; 0.046875] \times [0.0375; 0.0625] \times [0.0172; 0.0275$$

The jacobian matrix $J_G(x)$ is given by

$$\begin{pmatrix} \frac{\alpha}{8}\sin\cdots & -\frac{\alpha}{16}\sin\cdots & 0 & 0 \\ \frac{9\alpha}{64}\cos\cdots & 0 & \frac{3\alpha}{64}\cos\cdots & 0 \\ 0 & 0 & \frac{\alpha}{8}\sin\cdots & -\frac{\alpha}{16}\sin\cdots \\ 0 & 0 & \frac{6\alpha}{64}\sin\cdots & 0 \end{pmatrix}$$

and, therefore, $\|J_G(x)\|_\infty = \alpha \max\{\frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{32}\} = \frac{3\alpha}{16} \leq \frac{15}{16} < 1$.

What can we conclude?

- Since $G : \Omega \to \Omega$ is continuous and differentiable in the compact set Ω, and $\|J_G(x)\|_\infty \leq 1$ in Ω, the system has only and only one solution in this set and the fixed point method converges to this solution, for all initial approximation $x_0 \in \Omega$.
- If we take any $x_0 \in \mathbb{R}^n$ then $G(x_0) \in \Omega$, and so the convergence is guaranteed for any $x_0 \in \mathbb{R}^n$.
- We have the error bounds

$$\|z - x^{(n)}\|_\infty \leq (15/16)^n \|z - x^{(0)}\|_\infty \leq \frac{1}{16}(15/16)^n$$

$$\|z - x^{(n)}\|_\infty \leq \frac{15/16}{1 - 15/16} \|x^{(n)} - x^{(n-1)}\|_\infty$$

$$\|z - x^{(n)}\|_\infty \leq \frac{(15/16)^n}{1 - \frac{15}{16}} \|x^{(1)} - x^{(0)}\|_\infty$$

i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	$\|z - x^{(i)}\|_\infty$
1	0.0625	0.0625	0.0625	0.046875	0.315×10^{-1}
2	0.0614046	0.0319518	0.0607912	0.017169	0.123×10^{-2}
3	0.0601926	0.0314343	0.0594588	0.0167209	0.491×10^{-3}
4	0.0602878	0.0309125	0.0595855	0.0163703	0.308×10^{-4}
5	0.0602525	0.0309561	0.0595512	0.0164037	0.129×10^{-4}
6	0.0602582	0.0309413	0.059557	0.0163947	1.964×10^{-6}
7	0.0602569	0.0309437	0.0595559	0.0163962	4.522×10^{-7}
8	0.0602571	0.0309432	0.0595561	0.0163959	8.510×10^{-8}
21	0.062571	0.0309432	0.059556	0.0163959	6.592×10^{-17}

Table: Evolution of the computational error.

Newton's method for systems

If the Jacobian matrix of F is invertible, we have that

$$F(x) = 0 \Leftrightarrow -J_F^{-1}(x) \cdot F(x) = 0 \Leftrightarrow x = x - J_F^{-1}(x) \cdot F(x)$$

Newton's method for systems of equations

1. Take an initial guess x_0
2. For each $n > 0$ compute the solution y_n of $J_F(x_n)y_n = -F(x_n)$ and update $x_{n+1} = x_n + y_n$.
3. Stop when some criteria is satisfied.

Performance of Newton's method (same example)

i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	$\|z - x^{(i)}\|_\infty$
1	0.0625	0.0625	0.0625	0.046875	0.315×10^{-1}
2	0.0604482	0.0310313	0.059709	0.0164386	0.191×10^{-3}
3	0.0602571	0.0309433	0.0595561	0.016396	0.8329×10^{-7}
4	0.0602571	0.0309432	0.059556	0.0163959	0.1304×10^{-14}
5	0.0602571	0.0309432	0.059556	0.0163959	0.6938×10^{-17}

Table: Newton's method, evolution of the error