Mathematical Economics

Deterministic dynamic optimization

Discrete time

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Contents

1 Introduction

We introduce deterministic dynamic optimization problems and three methods for solving them.

Deterministic dynamic programming deals with finding deterministic sequences which verify some given conditions and which maximize (or minimise) a given intertemporal criterium.

1.1 Deterministic and optimal sequences

Consider the time set $\mathbb{T} = \{0, 1, \ldots, t, \ldots, T\}$ where T can be finite or $T = \infty$. We denote the value of variable at time t, by x_t. That is x_t is a mapping $x : \mathbb{T} \to \mathbb{R}$.

The timing of the variables differ: if x_t can be measured at **instant** t we call it a state variable, if u_t takes values in **period** t, which takes place between instants t and $t+1$ we call it a control variable.

Usually, stock variables (both prices and quantities) refer to instants and flow variables (prices and quantities) refer to periods.

A dynamic model is characterised by the fact that sequences have some form of intertemporal time-interaction. We distinguish intratemporal from intertemporal relations. Intratemporal, or period, relations take place within a single period and intertemporal relations involve trajectories.

A trajectory or path for state variables starting at $t = 0$ with the horizon $t = T$ is denoted by $x = \{x_0, x_1, \ldots, x_T\}$. We denote the trajectory starting at $t > 0$ by $x^t = \{x_t, x_{t+1}, \ldots, x_T\}$ and the trajectory up until time ${}^t x = \{x_0, x_1, \ldots, x_t\}$.

We consider two types of problems:

1. calculus of variations problems: feature sequences of state variables and evaluate these

sequences by an intertemporal objective function directly

$$J(x) = \sum_{t=0}^{T-1} F(t, x_t, x_{t+1})$$

2. optimal control problems: feature sequences of state and control variables, which are related by a sequence of intratemporal relations

$$x_{t+1} = g(x_t, u_t, t) \tag{1}$$

and evaluate these sequences by an intertemporal objective function over sequences (x, u)

$$J(x, u) = \sum_{t=0}^{T-1} f(t, x_t, u_t)$$

From equation (1) and the value of the state x_t at some points in time we could also determine an intertemporal relation[1].

In a deterministic dynamic model there is full information over the state x_t or the path x^t for any $t > s$ if we consider information at time s.

In general we have some conditions over the value of the state at time $t = 0$, x_0 and we may have other restrictions as well. The set of all trajectories x verifying some given conditions is denoted by $\mathcal{X}$. In optimal control problems the restrictions may involve both state and control sequence, x and u. In this case we denote the domain of all trajectories by $\mathcal{D}$

Usually $\mathcal{X}$, or $\mathcal{D}$, have infinite number of elements. Deterministic dynamic optimisation problems consist in finding the optimal sequences $x^\star \in \mathcal{X}$ (or $(x^\star, u^\star) \in \mathcal{D}$).

1.2 Some history

The calculus of variations problem is very old: Dido's problem, brachistochrone problem (Galileo), catenary problem and has been solved in some versions by Euler and Lagrange

[1] In economics the concept of sustainability is associated to meeting those types of intertemporal relations.

(XVII century) (see Liberzon (2012). The solution of the optimal control problem is due to Pontryagin et al. (1962). The dynamic programming method for solving the optimal control problem has been first presented by Bellman (1957).

1.3 Types of problems studied next

The problems we will study involve maximizing an intertemporal objective function (which is mathematically a **functional**) subject to some restrictions:

1. the **simplest calculus of variations problem**: we want to find a path $\{x_t\}_{t=0}^{T}$, where T, and both the initial and the terminal values of the state variable are known, $x_0 = \phi_0$ and $x_T = \phi_T$, such that it maximizes the functional $\sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$. Formally, the problem is: find a trajectory for the state of the system, $\{x_t^*\}_{t=0}^{T}$, that solves the problem
$$\max_{\{x_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t), \ s.t. \ x_0 = \phi_0, \ x_T = \phi_T$$
where ϕ_0, ϕ_T and T are given;

2. **calculus of variations problem with a free endpoint**: this is similar to the previous problem with the difference that the terminal state x_T is free. Formally:
$$\max_{\{x_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t), \ s.t. \ x_0 = \phi_0, \ x_T \text{ free}$$
where ϕ_0 and T are given;

3. the **optimal control problem with given terminal state**: we assume there are two types of variables, control and state variables, represented by u and x which are related by the difference equation $x_{t+1} = g(x_t, u_t)$. We assume that the initial and the terminal values of the state variable are known $x_0 = \phi_0$ and $x_T = \phi_T$ and we want to find an optimal trajectory joining those two states such that the functional

$\sum_{t=0}^{T-1} F(u_t, x_t, t)$ is maximized by choosing an optimal path for the control.

Formally, the problem is: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(u_t, x_t, t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \ldots T-1,\ x_0 = \phi_0,\ x_T = \phi_T$$

where ϕ_0, ϕ_T and T are given;

4. the **optimal control problem with free terminal state**: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(u_t, x_t, t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \ldots T-1,\ x_0 = \phi_0,\ x_T = \phi_T$$

where ϕ_0 and T are given.

5. in macroeconomics the **infinite time discounted optimal control problem** is the most common: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{\infty}$ and $\{x_t^*\}_{t=0}^{\infty}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t F(u_t, x_t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \ldots \infty,\ x_0 = \phi_0,$$

where $\beta \in (0, 1)$ is a discount factor and ϕ_0 is given. The terminal condition $\lim_{t\to} \eta^t x_t \geq 0$ is also frequently introduced, where $0 < \eta < 1$.

There are three methods for finding the solutions: (1) calculus of variations, for the first two problems, which is the reason why they are called calculus of variations problems, and (2) maximum principle of Pontriyagin and (3) dynamic programming, which can be used for all the five types of problems.

1.4 Some economic applications

The cake eating problem : let W_t be the size of a cake at instant t. If we eat C_t in period t, the size of the cake at instant $t+1$ will be $W_{t+1} = W_t - C_t$. We assume we know that the cake will last up until instant T. We evaluate the bites in the case by the intertemporal utility function featuring impatience, positive but decreasing marginal utility

$$\sum_{t=0}^{T-1} \beta^t u(C_t)$$

If the initial size of the cake is ϕ_0 and we want to consume it all until the end of period $T-1$ what will be the best eating strategy ?

The consumption-investment problem : let W_t be the financial wealth of a consumer at instant t. The intratemporal budget constraint in period t is

$$W_{t+1} = Y_t + (1+r)W_t - C_t,\ t = 0, 1, \ldots, T-1$$

where Y_t is the labour income in period t and r is the asset rate of return. The consumer has financial wealth W_0 initially. The consumer wants to determine the optimal consumption and wealth sequences $\{C_t\}_{t=0}^{T-1}$ and $\{W_t\}_{t=0}^{T}$ that maximises his intertemporal utility function

$$\sum_{t=0}^{T-1} \beta^t u(C_t)$$

where T can be finite or infinite.

The AK model growth model: let K_t be the stock of capital of an economy at time and consider the intratemporal aggregate constraint of the economy in period t

$$K_{t+1} = (1+A)K_t - C_t$$

where $F(K_t) = AK_t$ is the production function displaying constant marginal returns. Given the initial capital stock K_0 the optimal growth problem consists in finding the trajectory $\{K_t\}_{t=0}^{\infty}$ that maximises the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to a boundedness constraint for capital. The Ramsey (1928) model is a related model in which the production function displays decreasing marginal returns to capital.

2 Calculus of Variations

Calculus of variations problems were the first dynamic optimisation problems involving finding trajectories that maximize functionals given some restrictions. A functional is a function of functions, roughly. There are several types of problems. We will consider finite horizon (known terminal state and free terminal state) and infinite horizon problems.

2.1 The simplest problem

The **simplest calculus of variations problem** consists in finding a sequence that maximizes or minimizes a functional over the set of all trajectories $x \equiv \{x_t\}_{t=0}^{T}$, given initial and a terminal value for the state variable, x_0 and x_T.

Assume that $F(x', x)$ is continuous and differentiable in (x', x). The simplest problem of the calculus of variations is to find one (or more) optimal trajectory that maximizes the value functional

$$\max_{x} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \tag{2}$$

where the function $F(.)$ is called objective function

$$\text{subject to } x_0 = \phi_0 \text{ and } x_T = \phi_T \tag{3}$$

where ϕ_0 and ϕ_T are given.

Observe that the the upper limit of the sum should be consistent with the horizon of the problem T. In equation (2) the value functional is

$$\begin{aligned} V(\{x\}) &= \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \\ &= F(x_1, x_0, 0) + F(x_2, x_1, 1) + \ldots + F(x_t, x_{t-1}, t-1) + F(x_{t+1}, x_t, t) + \ldots \\ &\quad \ldots + F(x_T, x_{T-1}, T-1) \end{aligned}$$

because x_T is the terminal value of the state variable.

We denote the solution of the calculus of variations problem by $\{x_t^*\}_{t=0}^T$.

The **optimal value functional** is a number

$$V^* \equiv V(x^*) = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t) = \max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t).$$

Proposition 1. *(First order necessary condition for optimality)*
Let $\{x_t^*\}_{t=0}^T$ be a solution for the problem defined by equations (2) and (3). Then it verifies the Euler-Lagrange condition

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \dots, T-1 \tag{4}$$

and the initial and the terminal conditions

$$\begin{cases} x_0^* = \phi_0, & t = 0 \\ x_T^* = \phi_T, & t = T. \end{cases}$$

Proof. Assume that we know the optimal solution $x^* = \{x_t^*\}_{t=0}^T$. Therefore, we also know the optimal value functional $V(x^*) = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t)$. Consider an alternative candidate path as a solution of the problem, $\{x_t\}_{t=0}^{T-1}$ such that $x_t = x_t^* + \varepsilon_t$. In order to be admissible, it has to verify the restrictions of the problem. Then, we may choose $\varepsilon_t \neq 0$ for $t = 1, \dots, T-1$ and $\varepsilon_0 = \varepsilon_T = 0$. That is, the alternative candidate solution has the same initial and terminal values as the optimal solution, although following a different path. In this case the value function is

$$V(x) = \sum_{t=0}^{T-1} F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t).$$

where $\varepsilon_0 = \varepsilon_T = 0$. The variation of the value functional introduced by the perturbation $\{\varepsilon\}_{t=1}^{T-1}$ is

$$V(x) - V(x^*) = \sum_{t=0}^{T-1} F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t) - F(x_{t+1}^*, x_t^*, t).$$

If $F(.)$ is differentiable, we can use a first order Taylor approximation, evaluated along the trajectory $\{x_t^*\}_{t=0}^T$,

$$\begin{aligned}
V(x) - V(x^*) &= \frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_0}(x_0 - x_0^*) + \left(\frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_1} + \frac{\partial F(x_2^*, x_1^*, 1)}{\partial x_1}\right)(x_1 - x_1^*) + \ldots \\
&\quad \ldots + \left(\frac{\partial F(x_{T-1}^*, x_{T-2}^*, T-2)}{\partial x_{T-1}^*} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\right)(x_{T-1} - x_{T-1}^*) + \\
&\quad + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}(x_T - x_T^*) = \\
&= \left(\frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_1} + \frac{\partial F(x_2^*, x_1^*, 1)}{\partial x_1}\right)\varepsilon_1 + \ldots \\
&\quad \ldots + \left(\frac{\partial F(x_{T-1}^*, x_{T-2}^*, T-2)}{\partial x_{T-1}^*} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\right)\varepsilon_{T-1}
\end{aligned}$$

because $x_t - x_t^* = \varepsilon_t$ and $\varepsilon_0 = \varepsilon_T = 0$ Then

$$V(x) - V(x^*) = \sum_{t=1}^{T-1}\left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t}\right)\varepsilon_t. \tag{5}$$

If $\{x_t\}_{t=0}^{T-1}$ is an optimal solution then $V(x) - V(x^*) = 0$, which holds if (4) is verified. $\square$

Interpretation: equation (4) is an **intratemporal arbitrage condition** for period t. The optimal sequence has the property that at every period marginal benefits (from increasing one unit of x_t) are equal to the marginal costs (from sacrificing one unit of x_{t+1}):

Observations

- equation (4) is a non-linear difference equation of the second order: if we set

$$\begin{cases} y_{1,t} = & x_t \\ y_{2,t} = & x_{t+1} = y_{1,t+1}. \end{cases}$$

then the Euler Lagrange equation can be written as a planar equation in $\mathbf{y}_t = (y_{1,t}, y_{2,t})$

$$\begin{cases} y_{1,t+1} = y_{2,t} \\ \frac{\partial}{\partial y_{2,t}} F(y_{2,t}, y_{1,t}, t-1) + \frac{\partial}{\partial y_{2t}} F(y_{2,t+1}, y_{2,t}, t) = 0 \end{cases}$$

- if we have a minimum problem we have just to consider the symmetric of the value function
$$\min_{y} \sum_{t=0}^{T-1} F(y_{t+1}, y_t, t) = \max_{y} \sum_{t=0}^{T-1} -F(y_{t+1}, y_t, t)$$
- If $F(x, y)$ is concave then the necessary conditions are also sufficient.

Example 1: Let $F(x_{t+1}, x_t) = -(x_{t+1} - x_t/2 - 2)^2$, the terminal time $T = 4$, and the state constraints $x_0 = x_4 = 1$. Solve the calculus of variations problem.

Solution: If we apply the Euler-Lagrange equation we get a second order difference equation which is verified by the optimal solution

$$\frac{\partial}{\partial x_t}\left[-\left(x_t - \frac{x_{t-1}}{2} - 2\right)^2\right] + \frac{\partial}{\partial x_t}\left[-\left(x_{t+1} - \frac{x_t}{2} - 2\right)^2\right] = 0,$$

evaluated along $\{x_t^*\}_{t=0}^4$.

Then, we get

$$-2x_t^* + x_{t-1}^* + 4 + x_{t+1}^* - \frac{x_t^*}{2} - 2 = 0$$

If we introduce a time-shift, we get the equivalent Euler equation

$$x_{t+2}^* = \frac{5}{2}x_{t+1}^* - x_t^* - 2,\ t = 0, \ldots, T-2$$

which together with the initial condition and the terminal conditions constitutes a mixed initial-terminal value problem,

$$\begin{cases} x_{t+2}^* = \frac{5}{2}x_{t+1}^* - x_t^* - 2,\ t = 0, \ldots, 2 \\ x_0 = 1 \\ x_4 = 1. \end{cases} \tag{6}$$

In order to solve problem (6) we follow the method:

1. First, solve the Euler equation, whose solution is a function of two unknown constants (k_1 and k_2 next)

2. Second, we determine the two constants (k_1, k_2) by using the initial and terminal conditions.

First step: solving the Euler equation Next, we apply two methods for solving the Euler equation: (1) by direct methods, using equation (60) in the Appendix, or (2) solve it generally by transforming it to a first order difference equation system.

Method 1: applying the solution for the second order difference equation (60) Applying the results we derived for the second order difference equations we get:

$$x_t = 4 + \left(-\frac{1}{3}2^t + \frac{4}{3}\left(\frac{1}{2}\right)^t\right)(k_1 - 4) + \left(\frac{2}{3}2^t - \frac{2}{3}\left(\frac{1}{2}\right)^t\right)(k_2 - 4). \tag{7}$$

Method 2: general solution for the second order difference equation We follow the method:

1. First, we transform the second order equation into a planar equation by using the transformation $y_{1,t} = x_t$, $y_{2,t} = x_{t+1}$. The solution will be a known function of two arbitrary constants, that is $y_{1,t} = \varphi_t(k_1, k_2)$.

2. Second, we apply the transformation back the transformation $x_t = y_{1,t} = \varphi_t(k_1, k_2)$ which is function of two constants (k_1, k_2)

The equivalent planar system in $y_{1,t}$ and $y_{2,t}$ is

$$\begin{cases} y_{1,t+1} = y_{2,t} \\ y_{2,t+1} = \frac{5}{2}y_{2,t} - y_{1,t} - 2 \end{cases}$$

which is equivalent to a planar system of type $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}$ where

$$\mathbf{y}_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The solution of the planar system is $\mathbf{y}_t = \overline{\mathbf{y}} + \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}(k - \overline{\mathbf{y}})$ where $\overline{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ that is

$$\overline{y} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

and

$$\mathbf{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} 1/2 & 2 \\ 1 & 1 \end{pmatrix}, \ \mathbf{P}^{-1} = \begin{pmatrix} -2/3 & 4/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

Then

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}2^t & 2\left(\frac{1}{2}\right)^t \\ 2^t & \left(\frac{1}{2}\right)^t \end{pmatrix} \begin{pmatrix} -\frac{2}{3}(k_1 - 4) + \frac{4}{3}(k_2 - 4) \\ \frac{2}{3}(k_1 - 4) - \frac{1}{3}(k_2 - 4) \end{pmatrix}$$

If we substitute in the equation for $x_t = y_{1,t}$ and take the first element we have, again, the general solution of the Euler equation (7).

Second step: particular solution In order to determine the (particular) solution of the CV problem we take the general solution of the Euler equation (7), and determine k_1 and k_2 by solving the system $x_t|_{t=0} = 1$ and $x_t|_{t=4} = 1$:

$$4 + 1 \times (k_1 - 4) + 0 \times (k_2 - 4) = 1 \tag{8}$$

$$4 + \left(-\frac{1}{3}2^4 + \frac{4}{3}\left(\frac{1}{2}\right)^4\right)(k_1 - 4) + \left(\frac{2}{3}2^4 - \frac{2}{3}\left(\frac{1}{2}\right)^4\right)(k_2 - 4) = 1 \tag{9}$$

Then we get $k_1 = 1$ and $k_2 = 38/17$. If we substitute in the solution for x_t, we get

$$x_t^* = 4 - \frac{3}{17}2^t - \frac{48}{17}(1/2)^t$$

Therefore, the solution for the calculus of variations problem is the sequence

$$x^* = \{x^*\}_{t=0}^4 = \left\{1, \frac{38}{17}, \frac{44}{17}, \frac{38}{17}, 1\right\}.$$

Example 2: The cake eating problem Assume that there is a cake whose size at the beginning of period t is denoted by W_t and there is a muncher who wants to eat it until the beginning of period T. The initial size of the cake is $W_0 = \phi$ and, off course, $W_T = 0$ and the eater takes bites of size C_t at period t. The eater evaluates the utility of its bites through a logarithmic utility function and has a psychological discount factor $0 < \beta < 1$. What is the optimal eating strategy ?

Formally, the problem is to find the optimal paths $C^* = \{C_t^*\}_{t=0}^{T-1}$ and $W^* = \{W_t^*\}_{t=0}^{T}$ that solve the problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi,\ W_T = 0. \tag{10}$$

This problem can be transformed into the calculus of variations problem, because $C_t = W_t - W_{t+1}$,

$$\max_{W} \sum_{t=0}^{T} \beta^t \ln(W_t - W_{t+1}), \text{ subject to } W_0 = \phi,\ W_T = 0.$$

The Euler-Lagrange condition is:

$$-\frac{\beta^{t-1}}{W_{t-1}^* - W_t^*} + \frac{\beta^t}{W_t^* - W_{t+1}^*} = 0.$$

Then, the first order conditions are:

$$\begin{cases} W_{t+2}^* = (1+\beta)W_{t+1}^* - \beta W_t^*,\ t = 0, \dots T-2 \\ W_0 = \phi \\ W_T = 0 \end{cases}$$

In the appendix we find the solution of this linear scone order difference equation (see equation (56))

$$W_t^* = \frac{1}{1-\beta}\left(-\beta k_1 + k_2 + (k_1 - k_2)\beta^t\right),\ t = 0, 1 \dots, T \tag{11}$$

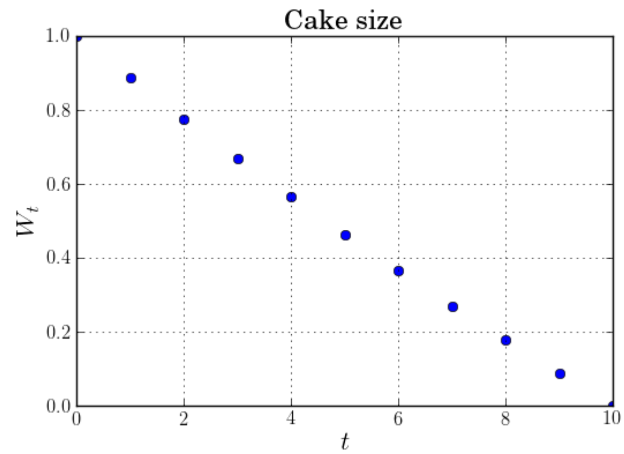

Figure 1: Solution to the cake eating problem with $T = 10$, $\phi_0 = 1$, $\phi_T = 0$ and $\beta = 1/1.03$

which depends on two arbitrary constants, k_1 and k_2. We can evaluate them by using the initial and terminal conditions

$$\begin{cases} W_0^* &= \frac{1}{1-\beta}\left(-\beta k_1 + k_2 + (k_1 - k_2)\right) = \phi \\ W_T^* &= -\beta k_1 + k_2 + (k_1 - k_2)\beta^T = 0. \end{cases}$$

Solving this linear system for k_1 and k_2, we get:

$$k_1 = \phi, \;\; k_2 = \frac{\beta - \beta^T}{1 - \beta^T}\phi$$

Therefore, the solution for the cake-eating problem $\{C^*\}$, $\{W^*\}$ is generated by

$$W_t^* = \left(\frac{\beta^t - \beta^T}{1 - \beta^T}\right)\phi, \; t = 0, 1, \dots T \tag{12}$$

and, as $C_t^* = W_t^* - W_{t+1}^*$

$$C_t^* = \left(\frac{1 - \beta}{1 - \beta^T}\right)\beta^t\phi, \; t = 0, 1, \dots T - 1. \tag{13}$$

2.2 Free terminal state problem

Now let us consider the problem

$$\max_{x} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

$$\text{subject to } x_0 = \phi_0 \text{ and } x_T \text{ free} \tag{14}$$

where ϕ_0 and T are given.

Proposition 2. *(Necessary condition for optimality for the free end point problem) Let $\{x_t^*\}_{t=0}^{T}$ be a solution for the problem defined by equations (2) and (14). Then it verifies the Euler-Lagrange condition*

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T-1 \tag{15}$$

and the initial and the transversality conditions

$$\begin{aligned} x_0^* &= \phi_0, \ t = 0 \\ \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} &= 0, \ t = T. \end{aligned} \tag{16}$$

Proof. Again we assume that we know $x^* = \{x_t^*\}_{t=0}^{T}$ and $V(x^*)$, and we use the same method as in the proof for the simplest problem. However, instead of equation (5) the variation introduced by the perturbation $\{\varepsilon_t\}_{t=0}^{T}$is

$$V(x) - V(x^*) = \sum_{t=1}^{T-2} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T$$

because $x_T = x_T^* + \varepsilon_T$ and $\epsilon_T \neq 0$ because the terminal state is not given. Then $V(x) - V(x^*) = 0$ if and only if the Euler and the transversality (16) conditions are verified. □

Condition (16) is called the **transversality condition**. Its meaning is the following: if the terminal state of the system is free, it would be optimal if there is no gain in changing the solution trajectory as regards the horizon of the program. If $\frac{\partial F(x^*_T, x^*_{T-1}, T-1)}{\partial x_T} > 0$ then we could improve the solution by increasing x^*_T (remember that the utiity functional is additive along time) and if $\frac{\partial F(x^*_T, x^*_{T-1}, T-1)}{\partial x_T} < 0$ we have an non-optimal terminal state by excess.

Example 1 (bis) Consider Example 1 and take the same objective function and initial state but assume instead that x_4 is free. In this case we have the terminal condition associated to the optimal terminal state,

$$2x^*_4 - x^*_3 - 4 = 0.$$

If we substitute the values of x_4 and x_3, from equation (7), we get the equivalent condition $-32 + 8k_1 + 16k_2 = 0$. This condition together with the initial condition, equation equation (8), allow us to determine the constants k_1 and k_2 as $k_1 = 1$ and $k_2 = 5/2$. If we substitute in the general solution, equation (7), we get $x_t = 4 - 3(1/2)^t$. Therefore, the solution for the problem is $\{1, 5/2, 13/4, 29/8, 61/16\}$, which is different from the path $\{1, 38/17, 44/17, 38/17, 1\}$ that we have determined for the fixed terminal state problem. □

However, in free endpoint problems we need sometimes an additional terminal condition in order to have a meaningful solution. To convince oneself, consider the following problem.

Cake eating problem with free terminal size . Consider the previous cake eating example where T is known but assume instead that W_T is free. The first order conditions from proposition (18) are

$$\begin{cases} W_{t+2} = (1+\beta)W_{t+1} - \beta W_t,\ t = 0, 1, \ldots, T-2 \\ W_0 = \phi \\ \frac{\beta^{T-1}}{W_T - W_{T-1}} = 0. \end{cases}$$

If we substitute the solution of the Euler-Lagrange condition, equation (11), the transversality condition becomes

$$\frac{\beta^{T-1}}{W_T - W_{T-1}} = \frac{\beta^{T-1}}{\beta^T - \beta^{T-1}} \frac{1-\beta}{k_1 - k_2} = \frac{1}{k_1 - k_2}$$

which can only be zero if $k_2 - k_1 = \infty$. If we look at the transversality condition, the last condition only holds if $W_T - W_{T-1} = \infty$, which does not make sense. □

The former problem is mispecified: the way we posed it it does not have a solution for bounded values of the cake.

One way to solve this, and which is very important in applications to economics is to introduce a terminal constraint.

2.3 Free terminal state problem with a terminal constraint

Consider the problem

$$\max_{\{x\}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

$$\text{subject to } x_0 = \phi_0 \text{ and } x_T \geq \phi_T \tag{17}$$

where ϕ_0, ϕ_T and T are given.

Proposition 3. *(Necessary condition for optimality for the free end point problem with terminal constraints)*
Let $\{x_t^*\}_{t=0}^T$ be a solution for the problem defined by equations (2) and (17). Then it verifies the Euler-Lagrange condition

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T-1 \tag{18}$$

and the initial and the transversality condition

$$\begin{cases} x_0^* = \phi_0, & t = 0 \\ \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T^*}(\phi_T - x_T^*) = 0, & t = T. \end{cases}$$

Proof. Now we write $V(\{x\})$ as a Lagrangian

$$V(\{x\}) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) + \lambda(\phi_T - x_T)$$

where λ is a Lagrange multiplier. Using again the variational method with $\epsilon_0 = 0$ and $\epsilon_T \neq 0$ the different between the perturbed candidate solution and the solution becomes

$$\begin{aligned} V(x) - V(x^*) &= \sum_{t=1}^{T-2} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \\ &\quad + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T + \lambda(\phi_T - x_T^* - \varepsilon_T) \end{aligned}$$

From the Kuhn-Tucker conditions, we have the conditions, regarding the terminal state,

$$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} - \lambda = 0, \ \lambda(\phi_T - x_T^*) = 0.$$

$\square$

The cake eating problem again Now, if we introduce the terminal condition $W_T \geq 0$, the first order conditions are

$$\begin{cases} W_{t+2}^* = (1+\beta)W_{t+1}^* - \beta W_t, \ t = 0, 1, \dots, T-2 \\ W_0^* = \phi \\ \frac{\beta^{T-1} W_T^*}{W_T^* - W_{T-1}^*} = 0. \end{cases}$$

If T is finite, the last condition only holds if $W_T^* = 0$, which means that it is optimal to eat all the cake in finite time. The solution is, thus formally, but not conceptually, the same as in the fixed endpoint case.

2.4 Infinite horizon problems

The most common problems in macroeconomics is the discounted infinite horizon problem.

We consider two problems, without or with terminal conditions.

No terminal condition

$$\max_{x} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_t) \tag{19}$$

where, $0 < \beta < 1$, $x_0 = \phi_0$ where ϕ_0 is given.

Proposition 4. *(Necessary condition for optimality for the infinite horizon problem) Let $\{x_t^*\}_{t=0}^{\infty}$ be a solution for the problem defined by equation (19). Then it verifies the Euler-Lagrange condition*

$$\frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} + \beta \frac{\partial F(x_{t+1}, x_t)}{\partial x_t} = 0,\ t = 0, 1, \ldots$$

and

$$\begin{cases} x_0^* = x_0, \\ \lim_{t\to\infty} \beta^{t-1} \frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} = 0, \end{cases}$$

Proof We can see this problem as a particular case of the free terminal state problem when $T = \infty$. Therefore the first order conditions were already derived. $\square$

With terminal conditions If we assume that $\lim_{t\to\infty} x_t = 0$ then the transversality condition becomes

$$\lim_{t\to\infty} \beta^t \frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} x_t^* = 0.$$

Exercise: the discounted infinite horizon cake eating problem The solution of the Euler-Lagrange condition was already derived as

$$W_t^* = \frac{1}{1-\beta} \left(-\beta k_1 + k_2 + (k_1 - k_2)\beta^t \right),\ t = 0, 1 \ldots, \infty$$

If we substitute in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t\to\infty} \beta^{t-1} \frac{\partial \ln(W_{t-1}^* - W_t^*)}{\partial W_t} = \lim_{t\to\infty} \beta^{t-1} (W_t^* - W_{t-1}^*)^{-1} = \lim_{t\to\infty} \frac{\beta^{t-1}}{\beta^t - \beta^{t-1}} \frac{1-\beta}{k_1 - k_2} = \frac{1}{k_2 - k_1}$$

which again ill-specified because the last equation is only equal to zero if $k_2 - k_1 = \infty$.

If we consider the infinite horizon problem with a terminal constraint $\lim_{t\to\infty} x_t \geq 0$ and substitute, in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t\to\infty} \beta^{t-1} \frac{\partial \ln(W_{t-1} - W_t)}{\partial W_t} W_t = \lim_{t\to\infty} \frac{W_t^*}{k_1 - k_2} = \frac{-\beta k_1 + k_2}{(1-\beta)(k_2 - k_1)}$$

because $\lim_{t\to\infty} \beta^t = 0$ as $0 < \beta < 1$. The transversality condition holds if and only if $k_2 = \beta k_1$. If we substitute in the solution for W_t, we get

$$W_t^* = \frac{k_1(1-\beta)}{1-\beta}\beta^t = k_1\beta^t,\ t = 0, 1 \ldots, \infty.$$

The solution verifie the initial condition $W_0 = \phi_0$ if and only if $k_1 = \phi_0$. Therefore the solution for the infinite horizon problem is $\{W_t^*\}_{t=0}^{\infty}$ where

$$W_t^* = \phi_0 \beta^t.$$

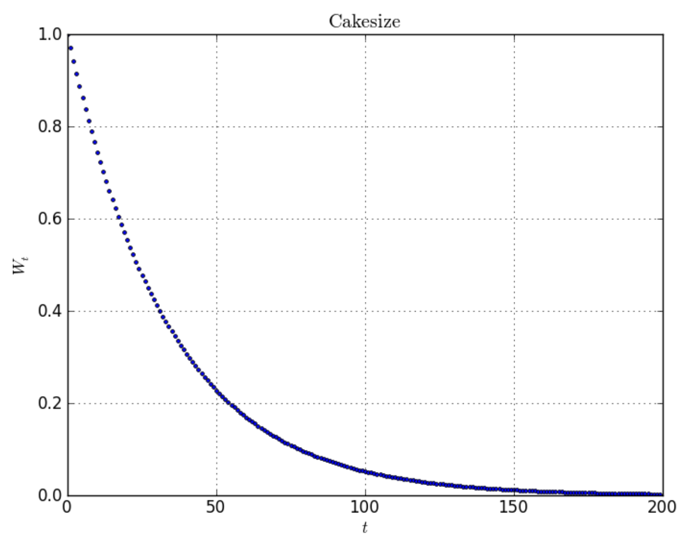

Figure 2: Solution for the cake eating problem with $T = \infty$, $\beta = 1/1.03$ and $\phi_0 = 1$

3 Optimal Control and the Pontriyagin's principle

The optimal control problem is a generalization of the calculus of variations problem. It involves two variables, the **control** and the **state** variables and consists in maximizing a functional over functions of the state and control variables subject to a difference equation over the state variable, which characterizes the system we want to control. Usually the initial state is known and there could exist or not additional terminal conditions over the state.

The trajectory (or orbit) of the state variable, $x \equiv \{x_t\}_{t=0}^{T}$, characterizes the state of a system, and the control variable path $u \equiv \{u_t\}_{t=0}^{T}$ allows us to control its evolution.

3.1 The simplest problem

Let T be finite. The simplest optimal control problem consist in finding the optimal paths $(\{u^*\}, \{x^*\})$ such that the value functional is maximized by choosing an optimal control,

$$\max_{\{u\}} \sum_{t=0}^{T-1} f(x_t, u_t, t), \tag{20}$$

subject to the constraints of the problem

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T = \phi_T & t = T \end{cases} \tag{21}$$

where ϕ_0, ϕ_T and T are given.

We assume that certain conditions hold: (1) differentiability of f; (2) concavity of g and f; (3) regularity [2]

Define the **Hamiltonian function**

$$H_t = H(\psi_t, x_t, u_t, t) = f(x_t, u_t, t) + \psi_t g(x_t, u_t, t)$$

[2]That is, existence of sequences of $x = \{x_1, x_2, ..., x_T\}$ and of $\overline{u} = \{\overline{u}_1, \overline{u}_2, ..., \overline{u}_T\}$ satisfying $\overline{x}_{t+1} = \frac{\partial g}{\partial x}(x_t^0, u_t^0)\overline{x}_t + g(x_t^0, \overline{u}_t) - g(x_t^0, u_t^0)$.

where ψ_t is called the **co-state variable** and $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ is the co-state variable path.

The **maximized** Hamiltonian

$$H_t^*(\psi_t, x_t^*) = \max_u H_t(\psi_t, x_t, u_t)$$

is obtained by substituting in H_t the optimal control, $u_t^* = u^*(x_t, \psi_t)$.

Proposition 5. *(Maximum principle)*
If x^* and u^* are solutions of the optimal control problem (20)-(21) and if the former differentiability and regularity conditions hold, then there is a sequence $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ such that the following conditions hold

$$\begin{aligned}
\frac{\partial H_t^*}{\partial u_t} &= 0,\ t = 0, 1, \ldots, T-1 && (22)\\
\psi_t &= \frac{\partial H_{t+1}^*}{\partial x_{t+1}},\ t = 0, \ldots, T-1 && (23)\\
x_{t+1}^* &= g(x_t^*, u_t^*, t) && (24)\\
x_T^* &= \phi_T && (25)\\
x_0^* &= \phi_0 && (26)
\end{aligned}$$

Proof. Assume that we know the solution (u^*, x^*) for the problem. Then the optimal value of value functional is $V^* = V(x^*) = \sum_{t=0}^{T-1} f(x_t^*, u_t^*, t)$.

Consider the Lagrangean

$$\begin{aligned}
L &= \sum_{t=0}^{T-1} f(x_t, u_t, t) + \psi_t(g(x_t, u_t, t) - x_{t+1})\\
&= \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1}
\end{aligned}$$

where Hamiltonian function is

$$H_t = H(\psi_t, x_t, u_t, t) \equiv f(x_t, u_t, t) + \psi_t(g(x_t, u_t, t). \quad (27)$$

Define

$$G_t = G(x_{t+1}, x_t, u_t, \psi_t, t) \equiv H(\psi_t, , x_t, u_t, t) - \psi_t x_{t+1}.$$

Then

$$L = \sum_{t=0}^{T-1} G(x_{t+1}, x_t, u_t, \psi_t, t)$$

If we introduce again a variation as regards the solution $\{u^*, x^*\}_{t=0}^{T}$, $x_t = x_t^* + \epsilon_t^x$, $u_t = u_t^* + \epsilon_t^u$ and form the variation in the value function and apply a first order Taylor approximation, as in the calculus of variations problem,

$$L - V^* = \sum_{t=1}^{T-1} \left(\frac{\partial G_{t-1}}{\partial x_t} + \frac{\partial G_t}{\partial x_t} \right) \epsilon_t^x + \sum_{t=0}^{T-1} \frac{\partial G_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \frac{\partial G_t}{\partial \psi_t} \epsilon_t^{\psi}.$$

Then, get the optimality conditions

$$\begin{aligned}
&\frac{\partial G_t}{\partial u_t} = 0,\ t = 0, 1, \ldots, T-1 \\
&\frac{\partial G_t}{\partial \psi_t} = 0,\ t = 0, 1, \ldots, T-1 \\
&\frac{\partial G_{t-1}}{\partial x_t} + \frac{\partial G_t}{\partial x_t} = 0,\ t = 1, \ldots, T-1
\end{aligned}$$

where all the variables are evaluated at the optimal path.

Evaluating these expressions for the same time period $t = 0, \ldots, T-1$, we get

$$\begin{aligned}
\frac{\partial G_t}{\partial u_t} &= \frac{\partial H_t}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u} + \psi_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u} = 0, \\
\frac{\partial G_t}{\partial \psi_t} &= \frac{\partial H_t}{\partial \psi_t} - x_{t+1} = g(x_t^*, u_t^*, t) - x_{t+1} = 0,
\end{aligned}$$

which is an admissibility condition

$$\begin{aligned}
\frac{\partial G_t}{\partial x_{t+1}} + \frac{\partial G_{t+1}}{\partial x_{t+1}} &= \frac{\partial (H_t - \psi_t x_{t+1})}{\partial x_{t+1}} + \frac{\partial H_{t+1}}{\partial x_{t+1}} \\
&= -\psi_t + \frac{\partial f(x_{t+1}^*, u_{t+1}^*, t+1)}{\partial x} + \psi_{t+1} \frac{\partial g(x_{t+1}^*, u_{t+1}^*, t+1)}{\partial x} = 0.
\end{aligned}$$

Then, setting the expressions to zero, we get, equivalently, equations (22)-(26) □

This is a version of the **Pontriyagin's maximum principle**. The first order conditions define a mixed initial-terminal value problem involving a planar difference equation.

If $\partial^2 H_t/\partial u_t^2 \neq 0$ then we can use the inverse function theorem on the static optimality condition

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

to get the optimal control as a function of the state and the co-state variables as

$$u_t^* = h(x_t^*, \psi_t, t)$$

if we substitute in equations (23) and (24) we get a non-linear planar ode in (ψ, x), called the **canonical system**,

$$\begin{cases} \psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1), \psi_{t+1}, t+1) \\ x_{t+1}^* = g(x_t^*, h(x_t^*, \psi_t, t), t) \end{cases} \tag{28}$$

where

$$\frac{\partial H_{t+1}^*}{\partial x_{t+1}} = \frac{\partial f(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1)}{\partial x_{t+1}} + \psi_{t+1} \frac{\partial g(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1)}{\partial x_{t+1}}$$

The first order conditions, according to the Pontriyagin principle, are then constituted by the canonical system (29) plus the initial and the terminal conditions (25) and (26).

Alternatively, if the relationship between u and ψ is monotonic, we could solve condition $\partial H_t^*/\partial u_t = 0$ for ψ_t to get

$$\psi_t = q_t(u_t^*, x_t^*, t) = -\frac{\frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t}}{\frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t}}$$

and we would get an equivalent (implicit or explicit) canonical system in (u, x)

$$\begin{cases} q_t(u_t^*, x_t^*, t) = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, u_{t+1}^*, q_{t+1}(u_{t+1}^*, x_{t+1}^*, t+1), t+1) \\ x_{t+1}^* = g(x_t^*, u_t^*, t) \end{cases} \tag{29}$$

which is an useful representation if we could isolate u_{t+1}, which is the case in the next example.

Exercise: cake eating Consider again problem (10) and solve it using the maximum principle of Pontriyagin. The present value Hamiltonian is

$$H_t = \beta^t \ln(C_t) + \psi_t(W_t - C_t)$$

and from first order conditions from the maximum principle

$$\begin{cases} \frac{\partial H_t^*}{\partial C_t} = \beta^t (C_t^*)^{-1} - \psi_t = 0,\ t = 0, 1, \ldots, T-1 \\ \psi_t = \frac{\partial H_{t+1}^*}{\partial W_{t+1}} = \psi_{t+1},\ t = 0, \ldots, T-1 \\ W_{t+1}^* = W_t^* - C_t^*,\ t = 0, \ldots, T-1 \\ W_T^* = 0 \\ W_0^* = \phi. \end{cases}$$

From the first two equations we get an equation over C, $C_{t+1}^* \beta^t = \beta^{t+1} C_t^*$, which is sometimes called the Euler equation. This equation together with the admissibility conditions, lead to the canonical dynamic system

$$\begin{cases} C_{t+1}^* = \beta C_t^* \\ W_{t+1}^* = W_t^* - C_t^*,\ t = 0, \ldots, T-1 \\ W_T^* = 0 \\ W_0^* = \phi. \end{cases}$$

There are two methods to solve this mixed initial-terminal value problem: recursively or jointly.

First method: we can solve the problem recursively. First,we solve the Euler equation to get

$$C_t = k_1 \beta^t.$$

Then the second equation becomes

$$W_{t+1} = W_t - k_1 \beta^t$$

which has solution

$$W_t = k_2 - k_1 \sum_{s=0}^{t-1} \beta^s = k_2 - k_1 \frac{1-\beta^t}{1-\beta}.$$

In order to determine the arbitrary constants, we consider again the initial and terminal conditions $W_0 = \phi$ and $W_T = 0$ and get

$$k_1 = \frac{1-\beta}{1-\beta^T}\phi, \; k_2 = \phi$$

and if we substitute in the expressions for C_t^* and W_t^* we get the same result as in the calculus of variations problem, equations (13)-(12).

Second method: we can solve the canonical system as a planar difference equation system. The first two equations have the form $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t$ where

$$\mathbf{A} = \begin{pmatrix} \beta & 0 \\ -1 & 1 \end{pmatrix}$$

which has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \beta$ and the associated eigenvector matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1-\beta \\ 1 & 1 \end{pmatrix}.$$

The solution of the planar equation is of type $\mathbf{y}_t = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}k$

$$\begin{aligned} \begin{pmatrix} C_t^* \\ W_t^* \end{pmatrix} &= \frac{1}{1-\beta} \begin{pmatrix} 0 & 1-\beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta^t \end{pmatrix} \begin{pmatrix} -1 & 1-\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \\ &= \begin{pmatrix} k_1\beta^t \\ k_2 - k_1\frac{1-\beta^t}{1-\beta} \end{pmatrix}. \end{aligned}$$

3.2 Free terminal state

Again, let T be finite. This is a slight modification of the simplest optimal control problem which has the objective functional (20) subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \dots, T-1 \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{30}$$

where ϕ_0 is given.

The Hamiltonian is the same as in the former problem and the first order necessary conditions for optimality are:

Proposition 6. *(Maximum principle)*
If $\{x^*\}_{t=0}^T$ and $\{u^*\}_{t=0}^T$ are solutions of the optimal control problem (20)-(30) and if the former assumptions on f and g hold, then there is a sequence $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ such that for $t = 0, 1, ..., T-1$

$$\frac{\partial H_t^*}{\partial u_t} = 0, \; t = 0, 1, \dots, T-1 \tag{31}$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \dots, T-1 \tag{32}$$

$$x_{t+1}^* = g(x_t^*, u_t^*, t) \tag{33}$$

$$x_0^* = \phi_0 \tag{34}$$

$$\psi_{T-1} = 0 \tag{35}$$

Proof. The proof is similar to the previous case, but now we have for $t = T$

$$\frac{\partial G_{T-1}}{\partial x_T} = \psi_{T-1} = 0.$$

□

3.3 Free terminal state with terminal constraint

Again let T be finite and assume that the terminal value for the state variable is non-negative. This is another slight modification of the simplest e simplest optimal control problem which has the objective functional (20) subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \geq 0 & t = T \end{cases} \tag{36}$$

where ϕ_0 is given.

The Hamiltonian is the same as in the former problem and the first order necessary conditions for optimality are

Proposition 7. *(Maximum principle)*
If $\{x^*\}_{t=0}^T$ and $\{u^*\}_{t=0}^T$ are solutions of the optimal control problem (20)-(36) and if the former conditions hold, then there is a sequence $\psi = \{\psi_t\}_{t=0}^{T-1}$ such that for $t = 0, 1, ..., T-1$ satisfying equations (31)-(34) and

$$\psi_{T-1} x_T^* = 0 \tag{37}$$

The cake eating problem Using the previous result, the necessary conditions according to the Pontryiagin's maximum principle are

$$\begin{cases} C_t = \beta^t / \psi_t \\ \psi_t = \psi_{t+1} \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \psi_{T-1} = 0 \end{cases}$$

This is equivalent to the problem involving the canonical planar difference equation system

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \frac{\beta^{T-1}}{C_{T-1}} = 0 \end{cases}$$

whose general solution was already found. The terminal condition becomes

$$\frac{\beta^{T-1}}{C_{T-1}} = \frac{\beta^{T-1}}{\beta^{T-1}k_1} = \frac{1}{k_1}$$

which can only be zero if $k_1 = \infty$, which does not make sense.

If we solve instead the problem with the terminal condition $W_T \geq 0$, then the transversality condition is

$$\psi_{T-1}W_T = \beta^{T-1}\frac{W_T}{C_{T-1}} = 0$$

If we substitute the general solutions for C_t and W_t we get

$$\beta^{T-1}\frac{W_T}{C_{T-1}} = \frac{1}{1-\beta}\left[\frac{-k_1 + (1-\beta)k_2}{k_1} + \frac{k_1}{k_1}\beta^T\right]$$

which is equal to zero if and only if

$$k_2 = k_1\frac{1-\beta^T}{1-\beta}.$$

We still have one unknown k_1. In order to determine it, we substitute in the expression for W_t

$$W_t = k_1\frac{\beta^t - \beta^T}{1-\beta},$$

evaluate it at $t = 0$, and use the initial condition $W_0 = \phi$ and get

$$k_1 = \frac{1-\beta}{1-\beta^T}\phi.$$

Therefore, the solution for the problem is the same as we got before, equations (13)-(12).

3.4 The discounted infinite horizon problem

The discounted infinite horizon optimal control problem consist on finding (u^*, x^*) such that

$$\max_u \sum_{t=0}^{\infty} \beta^t f(x_t, u_t), \ 0 < \beta < 1 \tag{38}$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t) & t = 0, 1, \ldots \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{39}$$

where ϕ_0 is given.

Observe that the functions $f(.)$ and $g(.)$ are now autonomous, in the sense that time does not enter directly as an argument, but there is a discount factor β^t which weights the value of $f(.)$ along time.

The **discounted Hamiltonian** is

$$h_t = h(x_t, \eta_t, u_t) \equiv f(u_t, y_t) + \eta_t g(y_t, u_t) \tag{40}$$

where η_t is the discounted co-state variable.

It is obtained from the current value Hamiltonian as follows:

$$\begin{aligned} H_t &= \beta^t f(u_t, x_t) + \psi_t g(x_t, u_t) \\ &= \beta^t \left(f(u_t, y_t) + \eta_t g(y_t, u_t) \right) \\ &\equiv \beta^t h_t \end{aligned}$$

where the co-state variable (η) relates with the actualized co-state variable (ψ) as $\psi_t = \beta^t \eta_t$. The Hamiltonian h_t is independent of time in discounted autonomous optimal control problems. The maximized current value Hamiltonian is

$$h_t^* = \max_u h_t(x_t, \eta_t, u_t).$$

Proposition 8. *(Maximum principle)*
If $x^* = \{x^*\}_{t=0}^{\infty}$ and $\{u^*\}_{t=0}^{\infty}$ is a solution of the optimal control problem (38)-(39) and if the former regularity and continuity conditions hold, then there is a sequence $\{\eta\} = \{\eta_t\}_{t=0}^{\infty}$ such that the optimal paths verify

$$\frac{\partial h_t^*}{\partial u_t} = 0,\ t = 0, 1, \ldots, \infty \tag{41}$$

$$\eta_t = \beta \frac{\partial h_{t+1}^*}{\partial x_{t+1}},\ t = 0, \ldots, \infty \tag{42}$$

$$x_{t+1}^* = g(x_t^*, u_t^*, t) \tag{43}$$

$$\lim_{t\to\infty} \beta^t \eta_t = 0 \tag{44}$$

$$x_0^* = \phi_0 \tag{45}$$

Proof. Exercise. □

Again, if we have the terminal condition

$$\lim_{t\to\infty} x_t \geq 0$$

the transversality condition is

$$\lim_{t\to\infty} \beta^t \eta_t x_t^* = 0 \tag{46}$$

instead of (44).

The necessary first-order conditions are again represented by the system of difference equations. If $\partial^2 h_t / \partial u_t^2 \neq 0$ then we can use the inverse function theorem on the static optimality condition

$$\frac{\partial h_t^*}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t} + \eta_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

to get the optimal control as a function of the state and the co-state variables as

$$u_t^* = h(x_t^*, \eta_t)$$

if we substitute in equations (42) and (43) we get a non-linear autonomous planar difference equation in (η, x) (or (u, x), if the relationship between u and η is monotonic)

$$\begin{cases} \eta_t = \beta \left(\frac{\partial f(x^*_{t+1}, h(x^*_{t+1}, \eta_{t+1}))}{\partial x_{t+1}} + \eta_{t+1} \frac{\partial g(x^*_{t+1}, h(x^*_{t+1}, \eta_{t+1}))}{\partial x_{t+1}} \right) \\ x^*_{t+1} = g(x^*_t, h(x^*_t, \eta_t)) \end{cases}$$

plus the initial and the transversality conditions (44) and (45) or (46).

Exercise: the cake eating problem with an infinite horizon The discounted Hamiltonian is

$$h_t = \ln{(C_t)} + \eta_t (W_t - C_t)$$

and the f.o.c are

$$\begin{cases} C_t = 1/\eta_t \\ \eta_t = \beta \eta_{t+1} \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t \to \infty} \beta^t \eta_t W_t = 0 \end{cases}$$

This is equivalent to the planar difference equation problem

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t \to \infty} \beta^t \frac{W_t}{C_t} = 0 \end{cases}$$

If we substitute the solutions for C_t and W_t in the transversality condition, we get

$$\lim_{t \to \infty} \beta^t \frac{W_t}{C_t} = \lim_{t \to \infty} \frac{-k_1 + (1-\beta) k_2 + k_1 \beta^t}{(1-\beta) k_1} = \frac{-k_1 + (1-\beta) k_2}{(1-\beta) k_1} = 0$$

if and only if $k_1 = (1-\beta)k_2$. Using the same method we used before, we finally reach again the optimal solution

$$C_t^* = (1-\beta)\phi\beta^t,\ W_t^* = \phi\beta^t,\ t = 0, 1, \dots, \infty.$$

Exercise: the consumption-savings problem with an infinite horizon Assume that a consumer has an initial stock of financial wealth given by $\phi > 0$ and gets a financial return if s/he has savings. The intratemporal budget constraint is

$$W_{t+1} = (1+r)W_t - C_t,\ t = 0, 1, \dots$$

where $r > 0$ is the constant rate of return. Assume s/he has the intertemporal utility functional

$$J(C) = \sum_{t=0}^{\infty} \beta^t \ln(C_t),\ 0 < \beta = \frac{1}{1+\rho} < 1,\ \rho > 0$$

and that the non-Ponzi game condition holds: $\lim_{t\to\infty} W_t \geq 0$. What are the optimal sequences for consumption and the stock of financial wealth ?

We next solve the problem by using the Pontriyagin's maximum principle. The discounted Hamiltonian is

$$h_t = \ln(C_t) + \eta_t\left((1+r)W_t - C_t\right)$$

where η_t is the discounted co-state variable. The f.o.c. are

$$\begin{cases} C_t = 1/\eta_t \\ \eta_t = \beta(1+r)\eta_{t+1} \\ W_{t+1} = (1+r)W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \eta_t W_t = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = 0 \end{cases}$$

If we define and use the first two and the last equation

$$z_t \equiv \frac{W_t}{C_t}$$

we get a boundary value problem

$$\begin{cases} z_{t+1} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right) \\ \lim_{t\to\infty} \beta^t z_t = 0. \end{cases}$$

The difference equation for z_t has the general solution [3]

$$z_t = \left(k - \frac{1}{(1+r)(1-\beta)}\right)\beta^{-t} + \frac{1}{(1+r)(1-\beta)}.$$

We can determine the arbitrary constant k by using the transversality condition:

$$\begin{aligned} \lim_{t\to\infty} \beta^t z_t &= \lim_{t\to\infty} \beta^t \left[\left(k - \frac{1}{(1+r)(1-\beta)}\right)\beta^{-t} + \frac{1}{(1+r)(1-\beta)}\right] \\ &= k - \frac{1}{(1+r)(1-\beta)} + \lim_{t\to\infty} \beta^t \left(\frac{1}{(1+r)(1-\beta)}\right) = \\ &= k - \frac{1}{(1+r)(1-\beta)} = 0 \end{aligned}$$

which is equal to zero if and only if

$$k = \frac{1}{(1+r)(1-\beta)}.$$

[3] The difference equation is of type $x_{t+1} = ax_t + b$, where $a \neq 1$ and has solution

$$x_t = \left(k - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$

where k is an arbitrary constant.

Then, $z_t = 1/\left((1+r)(1-\beta)\right)$ is a constant. Therefore, as $C_t = W_t/z_t$ the average and marginal propensity to consume out of wealth is also constant, and

$$C_t^* = (1+r)(1-\beta)W_t.$$

If we substitute in the intratemporal budget constraint and use the initial condition

$$\begin{cases} W_{t+1}^* = (1+r)W_t^* - C_t^* \\ W_0^* = \phi \end{cases}$$

we can determine explicitly the optimal stock of wealth for every instant

$$W_t^* = \phi\left(\beta(1+r)\right)^t = \left(\frac{1+r}{1+\rho}\right)^t,\ t = 0, 1, \ldots, \infty$$

and

$$C_t^* = (1+r)(1-\beta)\left(\frac{1+r}{1+\rho}\right)^t,\ t = 0, 1, \ldots, \infty.$$

We readily see that the solution depends crucially upon the relationship between the rate of return on financial assets, r and the rate of time preference ρ:

1. if $r > \rho$ then $\lim_{t\to\infty} W_t^* = \infty$: if the consumer is more patient than the market s/he optimally tends to have an abounded level of wealth asymptotically;

2. if $r = \rho$ then $\lim_{t\to\infty} W_t^* = \phi$: if the consumer is as patient as the market it is optimal to keep the level of financial wealth constant. Therefore: $C_t^* = rW_t = r\phi$;

3. if $r < \rho$ then $\lim_{t\to\infty} W_t^* = 0$: if the consumer is less patient than the market s/he optimally tends to end up with zero net wealth asymptotically.

The next figures illustrate the three cases

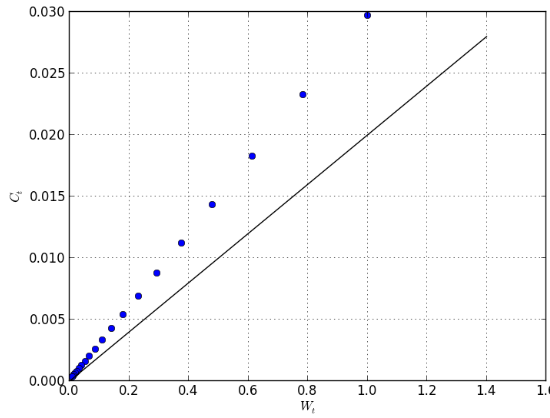

Figure 3: Phase diagram for the case in which $\phi > r$

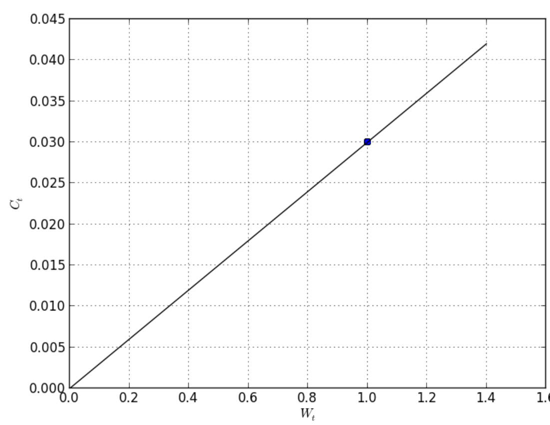

Figure 4: Phase diagram for the case in which $\phi = r$

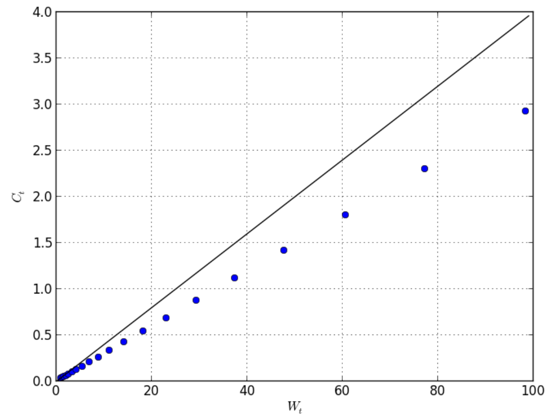

Figure 5: Phase diagram for the case in which $\phi < r$

Observe that although s/he may have an infinite level of wealth and consumption, asymptotically, the optimal value of the problem is bounded

$$
\begin{aligned}
J^* &= \sum_{t=0}^{\infty} \beta^t \ln (C_t^* = \\
&= \sum_{t=0}^{\infty} \beta^t \ln \left((1+r)(1-\beta) \left(\beta(1+r)\right)^t , \right) = \\
&= \sum_{t=0}^{\infty} \beta^t \ln \left((1+r)(1-\beta) \right) + \sum_{t=0}^{\infty} \beta^t \ln \left(\left(\beta(1+r)\right)^t \right) = \\
&= \frac{1}{1-\beta} \ln \left((1+r)(1-\beta) \right) + \ln \left(\beta(1+r) \right) \sum_{t=0}^{\infty} t\beta^t = \\
&= \frac{1}{1-\beta} \ln \left((1+r)(1-\beta) \right) + \frac{\beta}{(1-\beta)^2} \ln \left(\beta(1+r) \right)
\end{aligned}
$$

then

$$
J^* = \frac{1}{1-\beta} \ln \left(\left[(1+r)(1-\beta)^{1-\beta} \beta^{\beta} \right]^{1/(1-\beta)} \phi \right)
$$

which is always bounded.

4 Optimal control and the dynamic programming principle

Consider the discounted finite horizon optimal control problem which consists in finding (u^*, x^*) such that

$$\max_{u} \sum_{t=0}^{T} \beta^t f(x_t, u_t),\ 0 < \beta < 1 \tag{47}$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{48}$$

where ϕ_0 is given.

The principle of dynamic programming allows for an alternative method of solution.

According to the **Principle of the dynamic programming** (Bellman (1957)) an optimal trajectory has the following property: in the beginning of any period, take as given values of the state variable and of the control variables, and choose the control variables optimally for the rest of period. Apply this methods for every period.

4.1 The finite horizon problem

We start by the finite horizon problem, i.e. T finite.

Proposition 9. *Consider problem (47)-(48) with T finite. Then given an optimal solution the problem (x^*, u^*) satisfies the* ***Hamilton-Jacobi-Bellman equation***

$$V_{T-t}(x_t) = \max_{u_t} \left\{ f(x_t, u_t) + \beta V_{T-t-1}(x_{t+1}) \right\},\ t = 0, \ldots, T-1. \tag{49}$$

Proof. Define **value function** at time τ

$$V_{T-\tau}(x_\tau) = \sum_{t=\tau}^{T} \beta^{t-\tau} f(u_t^*, x_t^*) = \max_{\{u_t\}_{t=\tau}^{T}} \sum_{t=\tau}^{T} \beta^{t-\tau} f(u_t, x_t)$$

Then, for time $\tau = 0$ we have

$$\begin{aligned}
V_T(x_0) &= \max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t f(u_t, x_t) = \\
&= \max_{\{u_t\}_{t=0}^{T}} \big(f(x_0, u_0) + \beta f(x_1, u_1) + \beta^2 f(x_2, u_2) + \ldots \big) = \\
&= \max_{\{u_t\}_{t=0}^{T}} \left(f(x_0, u_0) + \beta \sum_{t=1}^{T} \beta^{t-1} f(x_t, u_t) \right) = \\
&= \max_{u_0} \left(f(x_0, u_0) + \beta \max_{\{u_t\}_{t=1}^{T}} \sum_{t=1}^{T} \beta^{t-1} f(x_t, u_t) \right)
\end{aligned}$$

by the principle of dynamic programming. Then

$$V_T(x_0) = \max_{u_0} \{ f(x_0, u_0) + \beta V_{T-1}(x_1) \}$$

We can apply the same idea for the value function for any time $0 \le t \le T$ to get the equation (49) which holds for feasible solutions, i.e., verifying $x_{t+1} = g(x_t, u_t)$ and given x_0. $\square$

Intuition: we transform the maximization of a functional into a recursive two-period problem. We solve the control problem by solving the HJB equation. To do this we have to find $\{V_T, \ldots, V_0\}$, through the recursion

$$V_{t+1}(x) = \max_{u} \{ f(x, u) + \beta V_t(g(x, u)) \} \tag{50}$$

Exercise: cake eating In order to solve the cake eating problem by using dynamic programming we have to determine a particular version of the Hamilton-Jacobi-Bellman equation (49). In this case, we get

$$V_{T-t}(W_t) = \max_{C_t} \{ \ln(C_t) + \beta V_{T-t-1}(W_{t+1}) \}, \; t = 0, 1, \ldots, T-1,$$

To solve it, we should take into account the restriction $W_{t+1} = W_t - C_t$ and the initial and terminal conditions.

We get the optimal policy function for consumption by deriving the right hand side for C_t and setting it to zero

$$\frac{\partial}{\partial C_t}\left\{\ln(C_t) + \beta V_{T-t-1}(W_{t+1})\right\} = 0$$

From this, we get the optimal policy function for consumption

$$C_t^* = \left(\beta V^{'}_{T-t-1}(W_{t+1})\right)^{-1} = C_t(W_{t+1}).$$

Then the HJB equation becomes

$$V_{T-t}(W_t) = \ln(C_t(W_{t+1})) + \beta V_{T-t-1}(W_{t+1}),\ t = 0, 1, \ldots, T-1 \tag{51}$$

which is a partial difference equation.

In order to solve it we make the **conjecture** that the solution is of the type

$$V_{T-t}(W_t) = A_{T-t} + \left(\frac{1-\beta^{T-t}}{1-\beta}\right)\ln(W_t),\ t = 0, 1, \ldots, T-1$$

where A_{T-t} is arbitrary. We apply the method of the undetermined coefficients in order to determine A_{T-t}.

With that trial function we have

$$C_t^* = \left(\beta V^{'}_{T-t-1}(W_{t+1})\right)^{-1} = \left(\frac{1-\beta}{\beta(1-\beta^{T-t-1})}\right)W_{t+1},\ t = 0, 1, \ldots, T-1$$

which implies. As the optimal cake size evolves according to $W_{t+1} = W_t - C_t^*$ then

$$W_{t+1} = \left(\frac{\beta-\beta^{T-t}}{1-\beta^{T-t}}\right)W_t. \tag{52}$$

which implies

$$C_t^* = \left(\frac{1-\beta}{1-\beta^{T-t}}\right)W_t,\ t = 0, 1, \ldots, T-1.$$

This is the same optimal policy for consumption as the one we got when we solve the problem by the calculus of variations technique. If we substitute back into the equation (51) we get an equivalent HJB equation

$$\begin{aligned} A_{T-t} &+ \left(\frac{1-\beta^{T-t}}{1-\beta}\right)\ln W_t = \\ &= \ln\left(\frac{1-\beta}{1-\beta^{T-t}}\right) + \ln W_t + \beta\left\{A_{T-t-1} + \left(\frac{1-\beta^{T-t-1}}{1-\beta}\right)\left[\ln\left(\frac{\beta-\beta^{T-t}}{1-\beta^{T-t}}\right) + \ln W_t\right]\right\} \end{aligned}$$

As the terms in $\ln W_t$ cancel out, this indicates (partially) that our conjecture was right. Then, the HJB equation reduces to the difference equation on A_t, the unknown term:

$$A_{T-t} = \beta A_{T-t-1} + \ln\left(\frac{1-\beta}{1-\beta^{T-t}}\right) + \left(\frac{\beta-\beta^{T-t}}{1-\beta}\right)\ln\left(\frac{\beta-\beta^{T-t}}{1-\beta^{T-t}}\right)$$

which can be written as a non-homogeneous difference equation, after some algebra,

$$A_{T-t} = \beta A_{T-t-1} + z_{T-t} \tag{53}$$

where

$$z_{T-t} \equiv \ln\left(\left(\frac{1-\beta}{1-\beta^{T-t}}\right)^{\frac{1-\beta^{T-t}}{1-\beta}}\left(\frac{\beta-\beta^{T-t}}{1-\beta}\right)^{\frac{\beta-\beta^{T-t}}{1-\beta}}\right)$$

In order to solve equation (53), we perform the change of coordinates $\tau = T - t$ and oberve that $A_{T-T} = A_0 = 0$ because the terminal value of the cake should be zero. Then, operating by recursion, we have

$$\begin{aligned} A_\tau &= \beta A_{\tau-1} + z_\tau = \\ &= \beta\left(\beta A_{\tau-2} + z_{\tau-1}\right) + z_\tau = \beta^2 A_{\tau-2} + z_\tau + \beta z_{\tau-1} = \\ &= \ldots \\ &= \beta^\tau A_0 + z_\tau + \beta z_{\tau-1} + \ldots + \beta^\tau z_0 \\ &= \sum_{s=0}^{\tau} \beta^s z_{\tau-s}. \end{aligned}$$

Then

$$A_{T-t} = \sum_{s=0}^{T-t} \beta^s \ln \left(\left(\frac{1-\beta}{1-\beta^{T-t-s}} \right)^{\frac{1-\beta^{T-t-s}}{1-\beta}} \left(\frac{\beta-\beta^{T-t-s}}{1-\beta} \right)^{\frac{\beta-\beta^{T-t-s}}{1-\beta}} \right).$$

If we use terminal condition $A_0 = 0$, then the solution of the HJB equation is, finally,

$$\begin{aligned} V_{T-t}(W_t) &= \ln \left(\prod_{s=0}^{T-t} \left(\frac{1-\beta}{1-\beta^{T-t-s}} \right)^{\frac{\beta^s-\beta^{T-t}}{1-\beta}} \left(\frac{\beta-\beta^{T-t-s}}{1-\beta} \right)^{\frac{\beta^{s+1}-\beta^{T-t}}{1-\beta}} \right) + \\ &+ \left(\frac{1-\beta^{T-t}}{1-\beta} \right) \ln(W_t), \; t = 0, 1, \ldots, T-1 \end{aligned} \tag{54}$$

We already determined the optimal policy for consumption (we really do not need to determine the term A_{T-t} if we only need to determine the optimal consumption)

$$C_t^* = \left(\frac{1-\beta}{1-\beta^{T-t}} \right) W_t = \left(\frac{1-\beta}{1-\beta^{T}} \right) \beta^t \phi, \; t = 0, 1, \ldots, T-1,$$

because, in equation (52) we get

$$\begin{aligned} W_t &= \beta \left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-1)}} \right) W_{t-1} = \\ &= \beta \left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-1)}} \right) \beta \left(\frac{1-\beta^{T-(t-1)}}{1-\beta^{T-(t-2)}} \right) W_{t-2} = \beta^2 \left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-2)}} \right) W_{t-2} = \\ &= \ldots \\ &= \beta^t \left(\frac{1-\beta^{T-t}}{1-\beta^{T}} \right) W_0 \end{aligned}$$

and $W_0 = \phi$.

4.2 The infinite horizon problem

For the infinite horizon discounted optimal control problem, the limit function $V = \lim_{j\to\infty} V_j$ is independent of j so the Hamilton Jacobi Bellman equation becomes

$$V(x) = \max_u \{ f(x,u) + \beta V[g(x,u)] \} = \max_u H(x,u)$$

Properties of the value function: it usually hard to get the properties of $V(.)$. In general continuity is assured but not differentiability (this is a subject for advanced courses on DP, see Stokey and Lucas (1989)).

If some regularity conditions hold, we may determine the optimal control through the **optimality condition**

$$\frac{\partial H(x,u)}{\partial u} = 0$$

if $H(.)$ is C^2 then we get the **policy function**

$$u^* = h(x)$$

which gives an optimal rule for changing the optimal control, given the state of the economy. If we can determine (or prove that there exists such a relationship) then we say that our problem is **recursive**.

In this case the HJB equation becomes a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta V[g(x, h(x))].$$

Solving the HJB: means finding the value function $V(x)$. Methods: analytical (in some cases exact) and mostly numerical (value function iteration).

Exercise: the cake eating problem with infinite horizon Now the HJB equation is

$$V(W) = \max_C \left\{ \ln(C) + \beta V(\tilde{W}) \right\},$$

where $\tilde{W} = W - C$. We say we solve the problem if we can find the unknown function $V(W)$.

In order to do this, first, we find the policy function $C^* = C(W)$, from the optimality condition

$$\frac{\partial \left\{ \ln(C) + \beta V(W-C) \right\}}{\partial C} = \frac{1}{C} - \beta V^{'}(W-C) = 0.$$

Then

$$C^* = \frac{1}{\beta V'(W - (C))},$$

which, if V is differentiable, yields $C^* = C(W))$.

Then $\tilde{W} = W - C^{'}(W) = \tilde{W}(W)$ and the HJB becomes a functional equation

$$V(W) = \ln\left(C^*(W)\right) + \beta V[\tilde{W}(W)].$$

Next, we try to solve the HJB equation by introducing a trial solution

$$V(W) = a + b\ln(W)$$

where the coefficients a and b are unknown, but we try to find them by using the **method of the undetermined coefficients**.

First, observe that

$$\begin{aligned} C &= \frac{1}{1+b\beta}W \\ \tilde{W} &= \frac{b\beta}{1+b\beta}W \end{aligned}$$

Substituting in the HJB equation, we get

$$a + b\ln\left(W\right) = \ln\left(W\right) - \ln\left(1+b\beta\right) + \beta\left(a + b\ln\left(\frac{b\beta}{1+b\beta}\right) + b\ln\left(W\right)\right),$$

which is equivalent to

$$(b(1-\beta)-1)\ln\left(W\right) = a(\beta-1) - \ln\left(1+b\beta\right) + \beta b\ln\left(\frac{b\beta}{1+b\beta}\right).$$

We can eliminate the coefficients of $ln(W)$ if we set

$$b = \frac{1}{1-\beta}.$$

Then the HJB equation becomes

$$0 = a(\beta-1) - \ln\left(\frac{1}{1-\beta}\right) + \frac{\beta}{1-\beta}\ln\left(\beta\right)$$

then

$$a = \ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta).$$

Then the value function is

$$V(W) = \frac{1}{1-\beta}\ln(\chi W), \text{ where } \chi \equiv \left(\beta^{\beta}(1-\beta)^{1-\beta}\right)^{1/(1-\beta)}.$$

and $C^* = (1-\beta)W$, that is

$$C_t^* = (1-\beta)W_t,$$

which yields the optimal cake size dynamics as

$$W_{t+1}^* = W_t - C_t^* = \beta W_t^*$$

which has the solution, again, $W_t^* = \phi\beta^t$.

5 Bibliographic references

(Ljungqvist and Sargent, 2004, ch. 3, 4) (de la Fuente, 2000, ch. 12, 13)

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A Second order linear difference equations

A.1 Autonomous problem

Consider the homogeneous linear second order difference equation

$$x_{t+2} = a_1 x_{t+1} + a_0 x_t, \tag{55}$$

where a_0 and a_1 are real constants and $a_0 \neq 0$.

The solution is

$$x_t = \left(\frac{\lambda_1 - a_1}{\lambda_1 - \lambda_2} \lambda_1^t + \frac{a_1 - \lambda_2}{\lambda_1 - \lambda_2} \lambda_2^t \right) k_1 - \frac{(\lambda_1 - a_1)(\lambda_2 - a_1)}{a_0(\lambda_1 - \lambda_2)} \left(\lambda_1^t - \lambda_2^t \right) k_2 \tag{56}$$

where k_1 and k_2 are arbitrary constants and

$$\lambda_1 = \frac{a_1}{2} - \left[\left(\frac{a_1}{2} \right)^2 + a_0 \right]^{1/2} \tag{57}$$

$$\lambda_2 = \frac{a_1}{2} + \left[\left(\frac{a_1}{2} \right)^2 + a_0 \right]^{1/2} \tag{58}$$

Proof: We can transform equation (55) into an equivalent linear planar difference equation of the first order, If we set $y_{1,t} \equiv x_t$ and $y_{2,t} \equiv x_{t+1}$, and observe that $y_{1,t+1} = y_{2,t}$ and equation (55) can be written as $y_{2,t+1} = a_0 y_{1,t} + a_1 y_{2,t}$.

Setting

$$\mathbf{y}_t \equiv \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}, \ \mathbf{A} \equiv \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}$$

we have, equivalently the autonomous first order system

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t,$$

which has the unique solution

$$\mathbf{y}_t = \mathbf{P}\mathbf{\Lambda}^t \mathbf{P}^{-1}\mathbf{k}$$

where $\mathbf{P}$ and $\mathbf{\Lambda}$ are the eigenvector and Jordan form associated to $\mathbf{A}$, $\mathbf{k} = (k_1, k_2)^\top$ is a vector of arbitrary constants.

The eigenvalue matrix is

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and, because $a_0 \neq 0$ implies that there are no zero eigenvalues,

$$\mathbf{P} = \begin{pmatrix} (\lambda_1 - a_1)/a_0 & (\lambda_2 - a_1)/a_0 \\ 1 & 1 \end{pmatrix}.$$

As $x_t = y_{1,t}$ then we get equation (56).

A.2 Non-autonomous problem

Now consider the homogeneous linear second order difference equation

$$x_{t+2} = a_1 x_{t+1} + a_0 x_t + b \tag{59}$$

where a_0, a_1 and b are real constants and $a_0 \neq 0$.

Case: $1 - a_1 - a_0 \neq 0$ If $1 - a_1 - a_0 \neq 0$ the general solution is

$$x_t = \bar{x} + \left(\frac{\lambda_1 - a_1}{\lambda_1 - \lambda_2} \lambda_1^t + \frac{a_1 - \lambda_2}{\lambda_1 - \lambda_2} \lambda_2^t \right) (k_1 - \bar{x}) - \frac{(\lambda_1 - a_1)(\lambda_2 - a_1)}{a_0(\lambda_1 - \lambda_2)} \left(\lambda_1^t - \lambda_2^t \right) (k_2 - \bar{x}) \tag{60}$$

where

$$\bar{x} = \frac{b}{1 - a_0 - a_1}$$

is the steady state of equation (59).

Proof: If we define $z_t \equiv x_t - \bar{x}$ then we get an equivalent system $\mathbf{y}_{t+1} - \bar{\mathbf{y}} = \mathbf{A}(\mathbf{y}_t - \bar{\mathbf{y}})$, where $\bar{\mathbf{y}} = (\bar{x}, \bar{x})^\top$ which has solution $\mathbf{y}_t - \bar{\mathbf{y}} = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}(\mathbf{k} - \bar{\mathbf{y}})$.

Case: $1 - a_1 - a_0 = 0$ If $1 - a_1 - a_0 = 0$ then the general solution of equation (59)

$$x_t = k_1 + k_2(a_0 - 1)^t + b\left(\frac{(2 - a_0)t - b}{(a_0 - 2)^2}\right)$$