LIBOR Convexity Adjustments for the Vasiček and Cox-Ingersoll-Ross models

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Abstract

In this paper we numerically implement some of the recent theoretical results concerning convexity adjustments derived within the affine term structure setup. The computation of the convexity adjustments in that setup is reduced to solving a system of ODES. Here we explore the Vasiček and Cox-Ingersoll-Ross models within LIBOR-inarrears and investigate how the convexity adjustments change with the model parameters. The two models reproduce the same behavior with the convexity adjustment showing up as an additive constant for maturity times > 5 years.

1 Introduction and Motivation

For fixed income markets, *convexity* has emerged as an intriguing and challenging notion. Taking this effect into account correctly could provide financial institutions with a competitive advantage. The idea underlying the notion of a convexity adjustment is quite intuitive and can be easily explained in the following terms. Many fixed income products are non-standard with respect to aspects such as the timing, the currency or the rate of payment. This leads to complex pricing formulas, many of which are hard to obtain in closed-form. Examples of such products include in-arreas or in-advance products, quanto products, CMS products, or equity swaps, among others. Despite their non-standard features, these products are quite similar to plain vanilla ones whose price can either be directly obtained from the market or at least computed in closed-form. Their complexity can be understood as introducing some sort of bias into the pricing of plain vanilla instruments. That is, we may decide to use the price of plain products and adjust it somehow to account for the complexity of non-standard products. This adjustment is what is known as convexity adjustment.

Under most stochastic interest rate setups convexity adjustments cannot be computed in closed-form and market practice is to use add-hoc rules or approximations when computing them. See, for instance, [1, 2, 3, 4, 5, 6]. Most of the times one has no clue on how big this approximation error may be although there is the hope convexity adjustments would be of a different order of magnitude, when compared to market prices, making all errors negligible.

In this paper we focus on timming adjustments and, in particular, on what we define to be LIBOR in-arrears adjustments (LIA adjustments). In [7] it was shown that, in any affine term structure setting, LIBOR adjustments can be obtained in closed-form, up to the solution of a system of ODEs. Here and for the popular models of Vasiček [8] and Cox-Ingersoll-Ross [9] models we numerically solved the necessary systems of ODEs and show, for a reasonable range of parameter values, convexity adjustments may be substantial in terms of market quotes. This undermines some of the market practices. Trough numerical experiments we find out and discuss term structure shapes for LIA convexity adjustments.

The paper is organized as follows. In Section 2 we set the notation, give the definitions and write the equations to be investigated. Details and proofs will be omitted as they can be found in [7]. In Section 3 we describe the Vasiček model and show the numerical results for the convexity adjustment LIA computed within the model. The exact analytical convexity adjustment for a particular solution of LIBOR in arrears is derived and analyzed. The Cox-Ingersoll-Ross model is discussed in Section 4. Finally, in Section 5 we resume our results and conclude.

2 Forward Convexity Adjustments for Affine Term Structure Models - the forward LIBOR in arrears case

In this section we resume the main results of [7] which are relevant for this work. The reader can find the details in the above cited work.

Assuming a stochastic interest rate setting, the no arbitrage price $\pi(t, \Phi)$, at time t, of a derivative paying $\Phi(T)$ at maturity T is given by

$$\pi(t,\Phi) \;=\; \mathbb{E}_t^Q \left[e^{-\int_t^T r_u \, du} \, \Phi(T) \right] \;=\; p(t,T) \; \mathbb{E}_t^T \left[\Phi(T) \right] \,, \tag{1}$$

where $p(t,T)$ is the price at time t of a pure discount bond with maturity T and $\mathbb{E}_t^Q[\cdot]$, $\mathbb{E}_t^T[\cdot]$ denotes the expectation under the risk neutral and the T-forward measure, respectively, conditional on the information available at time t. If the payoff is a T-forward martingale, then $\mathbb{E}_t^T \left[\Phi(T)\right] = \Phi(t)$. Unfortunately, this is not the case in most situations. Nonetheless, it may happen that the payoff is a martingale under a different measure. Let us denote this new martingale measure by $\mathbb{Q}^\Phi$, to stress its payoff dependency. In that case, it follows $\Phi(t) = \mathbb{E}_t^\Phi \left[\Phi(T)\right]$ and we can then *define* the convexity adjustment CC^Φ

$$\mathbb{E}_t^T \left[\Phi(T)\right] = \Phi(t) + CC_t^\Phi(t) \,. \tag{2}$$

as the amount we need to adjust the $\mathbb{E}_t^T \left[\Phi(T)\right]$ expectation to correct for the fact the payoff is not a martingale under the T- forward measure.

The connection between measure changes and convexity adjustments was first pointed out in [10] and has recently been further exploit in [7] in the context of affine term structure models.

As described in [7], in this context, various convexity adjustments can be computed exactly, without the need of approximation assumptions. Here we focus on in-arrears LIBOR adjustments. In the remaining of this section we establish the necessary notation, formally define the key ingredients, and state the main result from [7] that we use.

2.1 Affine Term Structure Setup

All the convexity results in [7] apply to multivariate models. Although the particular instances we will focus on in Sections 3 and 4 are univariate, we introduce here the necessary matrix notation, as the proposed numerics apply to both cases.

Let $(Z_t)_{t\geq 0}$ denote an $\mathbb{R}^m$-valued stochastic process whose dynamics, under the risk neutral measure $\mathbb{Q}$, is given by

$$dZ_t = \alpha(t, Z_t)\, dt \;+\; \sigma(t, Z_t)\, dW_t\,, \tag{3}$$

where W is an n-dimensional standard Brownian motion, $\alpha : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ are such that

$$\alpha(t, z) \;=\; d(t) + E(t)\, z \tag{4}$$

$$\sigma(t, z)\, \sigma^*(t, z) \;=\; k_0(t) + \sum_{i=1}^{m} k_i(t)\, z_i \tag{5}$$

with smooth functions $d : \mathbb{R}_+ \mapsto \mathbb{R}^m$ and E, k_0, k_i, $i = 1, \cdots, m$ mapping $\mathbb{R}_+$ into $\mathbb{R}^{m\times m}$. Furthermore, let the risk-free short rate $(r_t)_{t\geq 0}$ be defined as

$$r(t, Z_t) = f(t) + g(t)^* Z_t \tag{6}$$

where $f : \mathbb{R}_+ \mapsto \mathbb{R}$ and $g : \mathbb{R}_+ \mapsto \mathbb{R}^m$ are smooth functions. Here and in the following we use * to denote transpose.

For affine term structure models, it is well-known bond prices can be written as

$$p(t, T) = e^{A(t,T)+B(t,T)^* Z_t} \tag{7}$$

with the deterministic functions A and B being the solutions of the ordinary differential equation system

$$\frac{\partial A}{\partial t} + d(t)^* B + \frac{1}{2} B^* k_0(t)\, B \;=\; f(t)\,, \tag{8}$$

$$\frac{\partial B}{\partial t} + E(t)^* B + \frac{1}{2} \overline{B}^* K(t)\, B \;=\; g(t)\,, \tag{9}$$

subject to the boundary conditions $A(T,T) = 0$ and $B(T,T) = 0$. A and B should always be evaluated at (t,T). E, d, k_0, are from (4)-(5) and f, g from (6) while

$$\bar{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \qquad K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix} . \tag{10}$$

2.2 LIBOR rates and LIA convexity

Let us define now some LIBOR rates. LIBOR rates are quoted in the market and are, thus, the natural underlying to interest rate derivatives. The *forward* LIBOR rate, $L(t,T,S)$ with $t \leq T \leq S$, is the discrete compounding interest rate that can be contracted at time t to vigor during the time interval $[T,S]$. The *spot* LIBOR rate $L(t,S)$ is the particular instance of the forward one where the application period starts at the contract date $(t = T)$. By definition we have $L(t,S) = L(t,t,S)$ for all $t \leq S$ and hence $L(T,S) = L(T,T,S)$. For further details we refer to [11].

Absence of arbitrage in financial markets impose a necessary relationship between LIBOR rates and the price of pure discount bond maturing at time T and S,

$$L(t,T,S) = \frac{1}{S-T}\,\frac{p(t,T) - p(t,S)}{p(t,S)}\,. \tag{11}$$

From the definition it follows that $L(t,T,S)$ is a martingale under the S-forward measure. Pricing a payment of $L(T,S) = L(T,T,S)$ due at time S is, therefore, straightforward

$$\begin{aligned} \mathbb{E}_t^Q\left[e^{-\int_t^S r(u)\,du} L(T,S)\right] &= p(t,S)\,\mathbb{E}_t^S\left[L(T,T,S)\right] \\ &= p(t,S)\,L(t,T,S)\ , \end{aligned} \tag{12}$$

as both $p(t,S)$ and $L(t,T,S)$ are known at time t and can be simply observed from market quotes.

Many interest rate derivatives involve, however, in-arrears payments. These are payments of the spot LIBOR rate before its application period is over. The most common in-arrears payment is when the LIBOR rate, $L(T,S)$, is due at the very beginning of its application period (at time T) instead of at the end (at time S). The value of such payment can no longer be observed in the market as

$$\begin{aligned} \mathbb{E}_t^Q\left[e^{-\int_t^S r(u)\,du} L(T,S)\right] &= p(t,T)\,\mathbb{E}_t^T\left[L(T,S)\right] \\ &\neq p(t,S)\,L(t,T,S)\,. \end{aligned} \tag{13}$$

Note the last equality does not hold because $L(t,T,S)$ is not a T-martingale but a S-martingale. Still, practitioners, when evaluating $\mathbb{E}_t^T\left[L(T,S)\right]$, wish to use what can be directly observed in the market, the forward LIBOR $L(t,T,S)$, and then “adjust it” by adding a term known as convexity adjustment and that we here define as the *LIA convexity*. The LIA convexity adjustment $CC^{LIA}(t,T,S)$ is defined as

$$\mathbb{E}_t^T\left[L(T,S)\right] = L(t,T,S) + CC^{LIA}(t,T,S)\,. \tag{14}$$

For affine term structure models, in [7] it was proved that this convexity term is given by

$$CC^{LIA}(t,T,S) = \frac{1}{S-T}\,\frac{p(t,T)}{p(t,S)}\,\left[e^{F(t,T,S)+G(t,T,S)^*Z_t} - 1\right], \tag{15}$$

where F and G are smooth deterministic functions of (t,T,S) and solve the following ordinary differential equation system

$$\begin{cases} \frac{\partial F}{\partial t} + [B(t,S)-B(t,T)]^*k_0(t)[B(t,T)-B(t,S)] + d(t)^*G - B(t,T)^*k_0(t)G \\ \qquad\qquad +\frac{1}{2}G^*k_0(t)G + [B(t,T)-B(t,S)]^*k_0(t)G = 0 \\ \frac{\partial G}{\partial t} + [\overline{B}(t,S)-\overline{B}(t,T)]^*K(t)[B(t,T)-B(t,S)] + E(t)^*G - \overline{B}(t,T)K(t)G \\ \qquad\qquad +\frac{1}{2}\overline{G}^*K(t)G + [\overline{B}(t,T)-\overline{B}(t,S)]^*k_0(t)G = 0 \end{cases} \tag{16}$$

subject to the boundary conditions

$$F(T,T,S) = 0 \qquad \text{and} \qquad G(T,T,S) = 0\,. \tag{17}$$

The expression in Equation (15) is exact as it does not rely on any approximation and gives us the necessary LIA convexity in almost closed-form. The use of "almost" here emphasizes the fact that for most affine term structure models the ODE system in (16)-(17) is a matrix equation system with possibly time dependent coefficient that must be numerically evaluated.

2.3 Numerically solving the LIA convexity

Our goal is to evaluate $CC^{LIA}(t,T,S)$ that requires solving (16) subject to the boundary conditions (17).

The differential equations in (16) are non-linear and are of Riccati type. In general, find an exact analytical solution is quite difficult if not impossible. Therefore, in order to evaluate the functions F and G, to compute the convexity adjustments, one has to rely on numerical procedures.

In the following we solve (16), subject to the boundary conditions (17), for two popular affine models the Vasiček model [8], in Section 3, and the Cox-Ingersoll-Ross model [9], see Section 4, using a standard fourth order Runge-Kutta integrator [12] and integrating backwards in time. A solution, satisfying the boundary condition (17), can be obtained starting the integration at $t = T$ and setting $F(t = T,T,S) = 0$ and

$G(t = T, T, S) = 0$. The affine models investigated are univariate and the ODE system in (16)-(17) reduces to solving scalar equations.

The numerical procedure was tested building two independent codes, by different authors, and comparing the output. Acceptable solutions mean that, up to machine precision, the output of the two codes provided the same functions.

For the simulations discussed here we took the year as unit of time. Further, we have investigated how the solution depends on the integration time step δt and conclude that, for our choice of parameters and for both models discussed, a $\delta t = 0.01$ does not introduce any bias on the numerical solution. All the simulations reported here use such a value for δt.

3 Convexity Adjustments for LIA - the Vasiček model

In the Vasiček model the instantaneous short rate r is the only state variable, i.e. $Z(t) = r(t)$, and its dynamics are given by the following stochastic differential equation

$$dr = (b - a\,r)\,dt \;+\; \sigma\,dW, \qquad \text{with} \qquad a > 0. \tag{18}$$

The bond prices can be written as

$$p(t,T) = e^{A(t,T)+B(t,T)r(t)}\,, \tag{19}$$

with

$$B(t,T) \;=\; \frac{1}{a}\left\{e^{-a(t-T)} - 1\right\}, \tag{20}$$

$$A(t,T) \;=\; \frac{B(t,T) - T + t}{a^2}\left(ab - \frac{\sigma^2}{2}\right) \;-\; \frac{\sigma^2 B^2(t,T)}{4a}\,. \tag{21}$$

The computation of function $A(t,T)$ and $B(t,T)$ can be found, for example, in [11].

For the Vasiček model, the ordinary differential equation system (16) defining the the convexity adjustments simplifies considerable. Indeed, the equation for G becomes

$$\frac{\partial G}{\partial t} - a\,G = 0\,, \tag{22}$$

whose solution is

$$G(t,T,S) = G_0\,e^{a\,t} + G_1(T,S) \tag{23}$$

with $G_1(T,S)$ being an arbitrary function of (T,S). The boundary condition on G, see equation (17), implies that

$$G(t,T,S) = G_0\,\left(e^{a\,t} - e^{a\,T}\right)\,. \tag{24}$$

On the other hand, the equation for F reads

$$\begin{aligned}\frac{\partial F}{\partial t} \;&+\; \sigma^2\left[B(t,S) - B(t,T)\right]\left[B(t,T) - B(t,S)\right] \\ &+\; \left\{b - \sigma^2 B(t,T)\right\}G \;+\; \frac{\sigma^2}{2}G^2 = 0\,.\end{aligned} \tag{25}$$

The Vasiček model does not constraint in anyway $G(t,T,S)$ and, therefore, $F(t,T,S)$. At most, the functions $G(t,T,S)$ and $F(t,T,S)$ can be computed looking at the market prices. From the point of view of the calculation of CC^{LIA} this is a problem of the model, i.e. one is not able to compute the convexity adjustment unless *a priori* hypothesis are made on G. For the Vasiček model, the system (16) together with the boundary conditions (17) is not complete, in the sense that it does allows a unique solution. Note that, for example, for the Cox-Ingersoll-Ross model (16) and (17) are able to define a unique solution - see the discussion in section 4.

Since our goal is to compute CC^{LIA} one has to further constraint G. We observe that the set of ordinary differential equations includes the particular solution $G(t,T,S) = 0$. Then, in the following, only the particular solution with $G(t,T,S) = 0$ will be considered.

For the particular solution where $G(t,T,S) = 0$, the differential equation that remains to be solved reads

$$\frac{\partial F}{\partial t} - \sigma^2 \left[B(t,S) - B(t,T)\right]^2 \;=\; 0\,. \tag{26}$$

In section 3.1 this equations is solved exactly. Here we proceed with the discussion of its numerical solution.

Equation (26) can be solved numerically using a standard fourth order Runge-Kutta integrator and integrating backwards in time. A solution, satisfying the boundary condition (17), can be obtained starting the integration at $t = T$ and setting $F(t = T,T,S) = 0$. The numerical procedure was tested building two independent codes, by different authors, and comparing the output. Acceptable solutions mean that, up to machine precision, the output of the two codes provided the same function.

For the numerical simulation one has to chose realistic values for the parameters a, b and the volatility σ. For the parameters a and b we rely on [13], where it was investigated how well the Vasiček model reproduce the market prices, and take as typical values $a = 0.7$ and $b = 0.05$ and an initial short rate $r = 5\%$. Moreover, without loss of generality, we set $t = 0$ and we introduce the following notation

$$S = T + \Delta_{ST} \tag{27}$$

to investigate different distances between T and S. Naturally, the bigger the Δ_{ST} the bigger is the LIA convexity.

The term structure of bond prices for different volatilities is reported in figure 1.

The function $F(0,T,T+\Delta_{ST})$, a first step to compute the convexity adjustment, is seen in Figure 2 for various Δ_{ST} and volatilities. The figure shows a universal F behavior. After a sharp rise, F becomes flat for T larger than $\sim 2-3$, depending on the value of Δ_{ST}. This means that for large maturity times, convexity corrections are essentially a function of the bond price, corrected by a multiplicative constant defined by the asymptotic value of F - see equation (15).

In what concerns the σ dependence, F increases with the volatility and the data in Figure 2 suggests a $F \propto \sigma^2$. This behavior with the volatility is confirmed by the investigation of the exact analytical solution of equation (26) - see Section 3.1. Indeed,

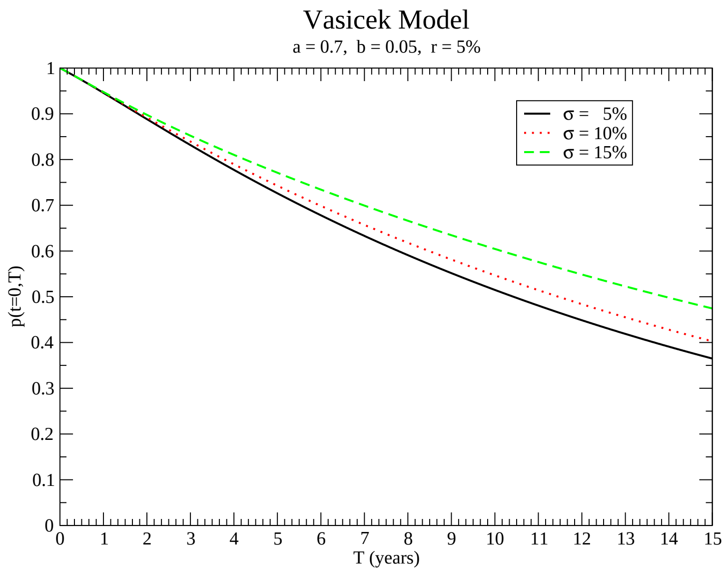

Figure 1: Bond price for the Vasiček model.

our numerical solution follows the predictions of the analytical solution discussed in section 3.1 and this good agreement gives further confidence in our results.

The LIA convexity adjustment are plotted in Figure 3. The corrections to $L(0, T, T + \Delta_{ST})$ are, for the model under discussion, positive and are always below the 10% level. Moreover, the adjustments increase both with the volatility and the difference between maturity time and payoff time, i.e. with Δ_{ST}. Similarly to the F, CC^{LIA} increases rapidly for small T, showing an almost constant behavior for T larger than ~ 3 years.

For completeness, in Figure 4 we show the LIA together with the $L(0, T, T + \Delta_{ST}) + CC^{LIA}(0, T, T + \Delta_{ST})$. The correction to LIA decreases with the volatility and, for large maturity times, they looks like adding a positive constant to the LIBOR rate.

3.1 CC^{LIA} - The Analytical Solution for the Vasiček Model

The solution of equation (22) given in (24) is a linear combination of exponentials. More, the differential equation for F, see equation (25), requires only $B(t, T)$ which is given by the sum of a constant with an exponential function. This means that, using standard integration methods, one can integrate (25) exactly. The solution of the full equation (25) is rather lengthly and it will not be given here. Instead, we will discuss the solution of (25) when $G_0 = 0$, the particular case solved numerically.

Setting $G = 0$ in (22), the ordinary differential equation simplifies into (26). For $B(t, T)$ given by (20), it follows that

$$\begin{aligned} F(t,T,S) \; = \; & \frac{\sigma^2}{2\,a^3} \; \left\{ 2 \left[e^{-a(2t-S-T)} - e^{-a(T-S)} \right] + \right. \\ & \left. + \left[1 - e^{-2\,a(t-T)} \right] + \left[e^{-2\,a(T-S)} - e^{-2\,a(t-S)} \right] \right\} \end{aligned} \qquad (28)$$

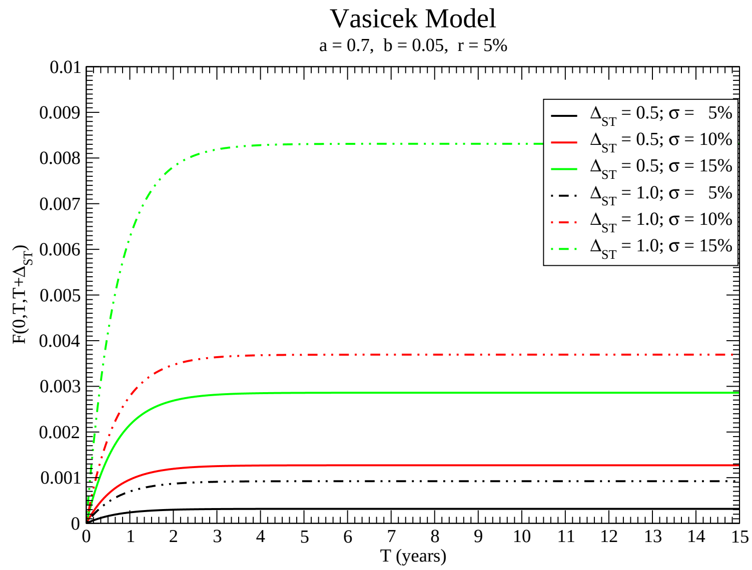

Figure 2: $F(0, T, T + \Delta_{ST})$ for the Vasiček model for various volatilities and different Δ_{ST}.

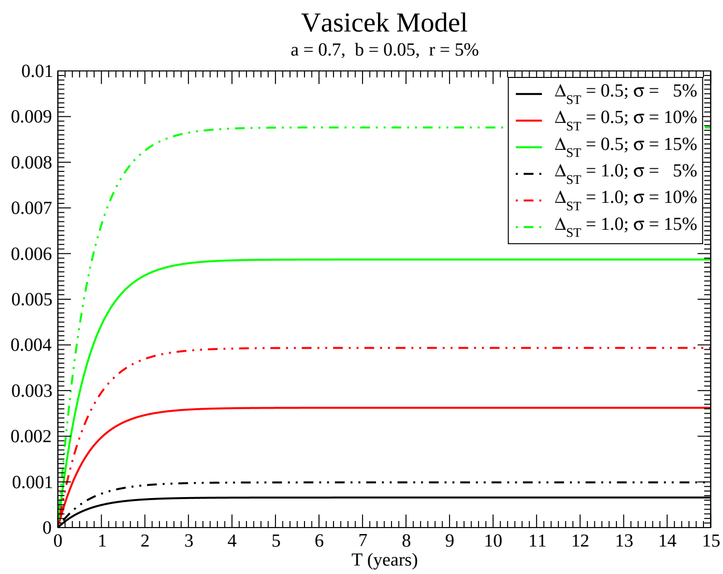

Figure 3: $CC^{LIA}(0, T, T + \Delta_{ST})$ for the same set of parameters as in Figure 2.

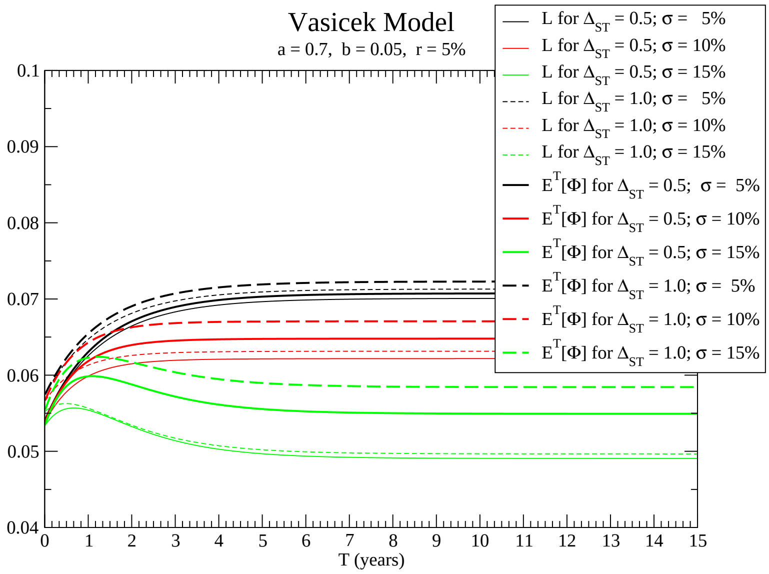

Figure 4: $L(0,T,T+\Delta_{ST})$ and $L(0,T,T+\Delta_{ST})+CC^{LIA}(0,T,T+\Delta_{ST})$ for the simulations parameters discussed in the text.

and

$$
\begin{aligned}
F(0,T,T+\Delta_{ST}) \;&=\; \tfrac{\sigma^2}{2\,a^3} \quad \Big\{ 2\,e^{a(2T+\Delta_{ST})} + e^{2a\Delta_{ST}} \;+1 \\
&\qquad\qquad -e^{2a(T+\Delta_{ST})} - e^{2aT} - 2e^{a\Delta_{ST}} \Big\}\,.
\end{aligned} \tag{29}
$$

From the definition of F and for small T, after expanding F as a Taylor series in T and keeping only the lowest order term, one can write

$$
F(0,T,T+\Delta_{ST}) = \frac{\sigma^2}{a^2}\left(1+e^{2\,a\,\Delta_{ST}} - 2\,e^{a\,\Delta_{ST}}\right)T + \mathcal{O}(T^2)\,. \tag{30}
$$

For the parameters used in the simulation, the coefficient multiplying T is a positive number. The corresponding linear behavior for short T is clearly seen in Figure 2. Furthermore, we have checked that our numerical solution follows the (30) for all ranges of T. This gives us further confidence in our numerics.

4 Convexity Adjustments for LIA - the Cox-Ingersoll-Ross model

Let us now discuss the LIA convexity adjustments in the context of the Cox-Ingersoll-Ross model. The short rate dynamics is given by the following stochastic differential equation

$$
dr = a(b-r)\,dt + \sigma\sqrt{r}\,dW\,. \tag{31}
$$

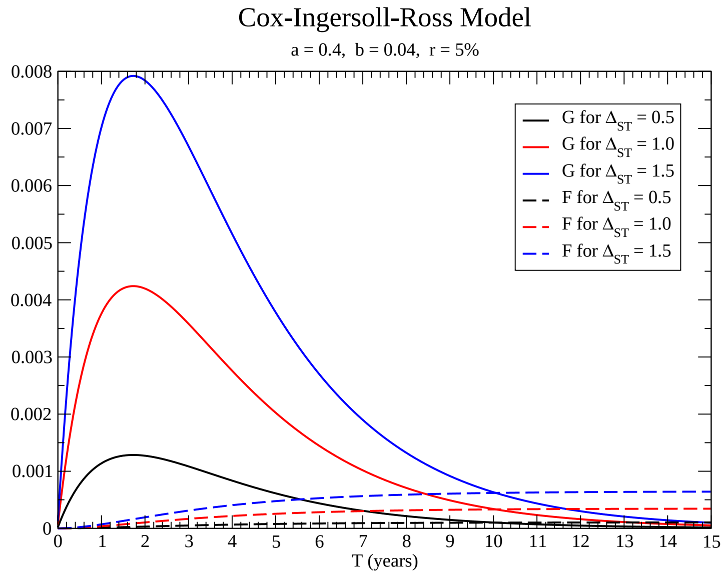

Figure 5: Typical $F(0,T,T+\Delta_{ST})$ and $G(0,T,T+\Delta_{ST})$ for different Δ_{ST} for the Cox-Ingersoll-Ross model.

The reader should be aware although one is using the same notation for the Vasiček and the Cox-Ingersoll-Ross model, the parameters a and b don't have exactly the same meaning - see equations (18) and (31).

For the Cox-Ingersol-Ross model the bond price can also be written as (19) with

$$\begin{aligned} A(t,T) &= \frac{2\,a\,b}{\sigma^2}\,\ln\left\{\frac{2\,\gamma\,e^{(a+\gamma)(T-t)/2}}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)\,+\,2\gamma}\right\}, \quad (32) \\ B(t,T) &= \frac{2\,\left(1-e^{\gamma(T-t)}\right)}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)\,+\,2\,\gamma} \quad (33) \end{aligned}$$

and where $\gamma=\sqrt{a^2+2\,\sigma^2}$. The system of differential equations for F and G now reads

$$\begin{cases} \frac{\partial F}{\partial t}\,+\,a\,b\,G\,=0\,, \\ \\ \frac{\partial G}{\partial t}\,+\,\sigma^2[B(t,S)-B(t,T)][B(t,T)-B(t,S)] \\ \qquad\qquad -\,\left[\sigma^2 B(t,S)+a\right]G+\frac{\sigma^2}{2}\,G^2=0\,. \end{cases} \quad (34)$$

This system together with the boundary conditions (17) define a unique solution for F and G, i.e. for the Cox-Ingersoll-Ross model the convexity adjustment can be computed without ambiguities.

The equations (34) were solved numerically following the some methodology used in the analysis of the Vasiček model. The differential equations were solved by backward integration starting at $t=T$ and setting $F(T,T,S+\Delta_{ST})=G(T,T,S+\Delta_{ST})=0$. As

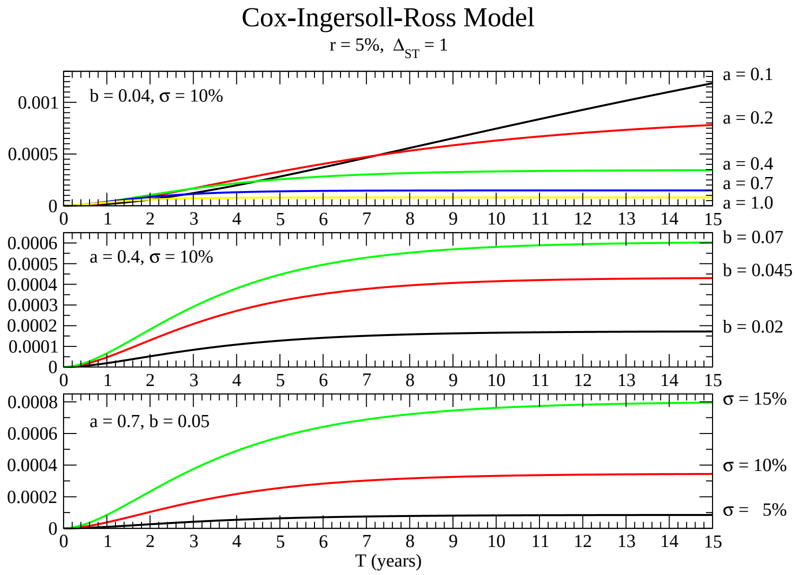

Figure 6: $F(0, T, T + \Delta_{ST})$ as a function of the model parameters.

before, for the integration step we used $\delta t = 0.01$. For the choice of parameters, as in the previous section we rely on [13].

Typical solutions for F and G are reported in Figure 5 for different Δ_{ST}. In all our simulations we have observed that G is much larger than F for small to medium maturities times, with F giving the larger contribution to the convexity adjustment in the long run. Furthermore, while G has a maximum at $T \sim 1 - 2$ years and then decreases monotonically, F always increase with the maturity time T.

The function $F(0, T, T + \Delta_{ST})$ is not independent of the parameters which defined the model. Indeed, our simulations show that F increases as T, b, σ and Δ_{ST} take larger values and when a becomes smaller. How F changes with $S - T$ is illustrated in Figure 5. The remaining dependences are shown in figure 6.

How $G(t, T, S)$ change with T, b, σ and Δ_{ST} is reported in Figures 5 and 7. The numerical simulations give a G blind to b. More, the maximum of G happens at the same T but the function increases as the volatility increases. The dependence of G on Δ_{ST} is similar, i.e. the maximum seems to be independent of Δ_{ST} and the maximum of G increases with Δ_{ST}. In what concerns the dependence with a, it was observed that smaller values of the parameter produce larger values of G for larger maturity times. Furthermore, for sufficient small a, G increases with from zero and seems to approach a plateau, i.e. for smaller a the function G mimics the behavior of F found for the Vasiček model.

The functions F and G are interesting in the sense that they are required to computed the LIA convexity adjustment defined in equation (15). From the discussion of the previous paragraphs, one can claim that CC^{LIA} is mainly given by G for short and medium maturity times, while for large T the correction CC^{LIA} is given essentially by

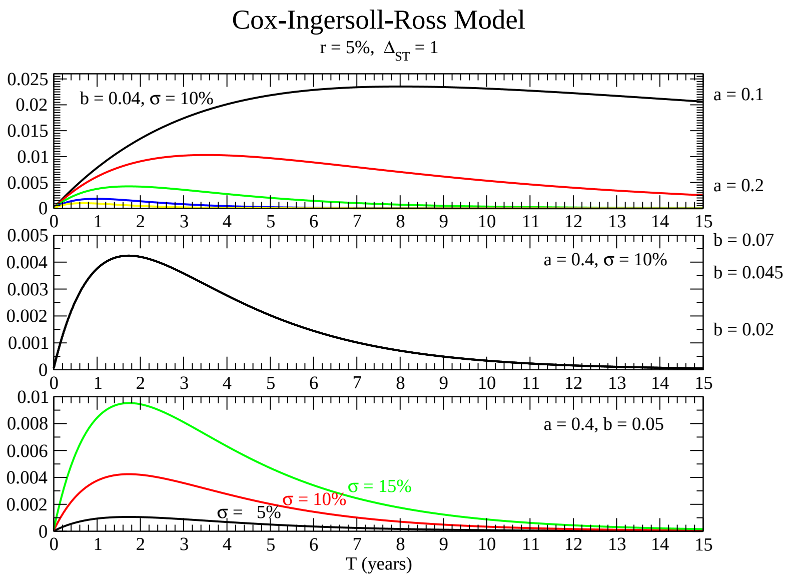

Figure 7: $G(0, T, T + \Delta_{ST})$ as a function of the model parameters. The curves shown in the first graph follow the same color code/a-value as in the previous figure.

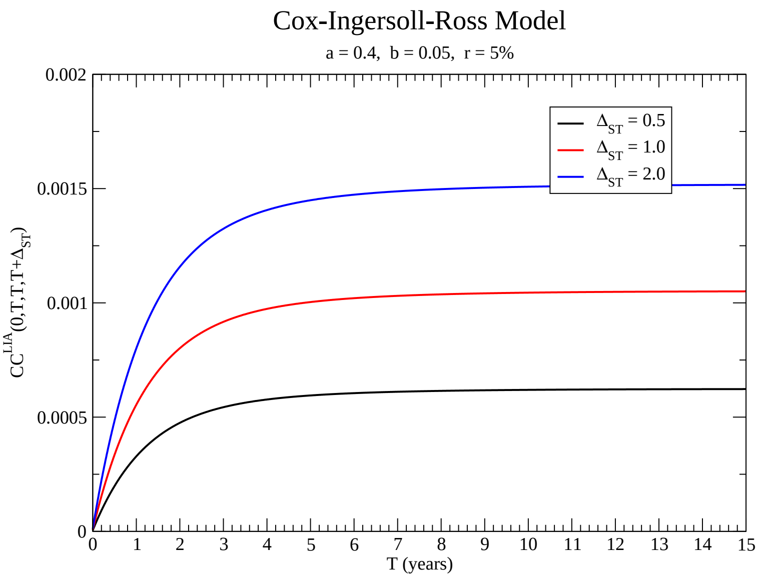

Figure 8: $CC^{LIA}(0, T, T + \Delta_{ST})$ for the various Δ_{ST}.

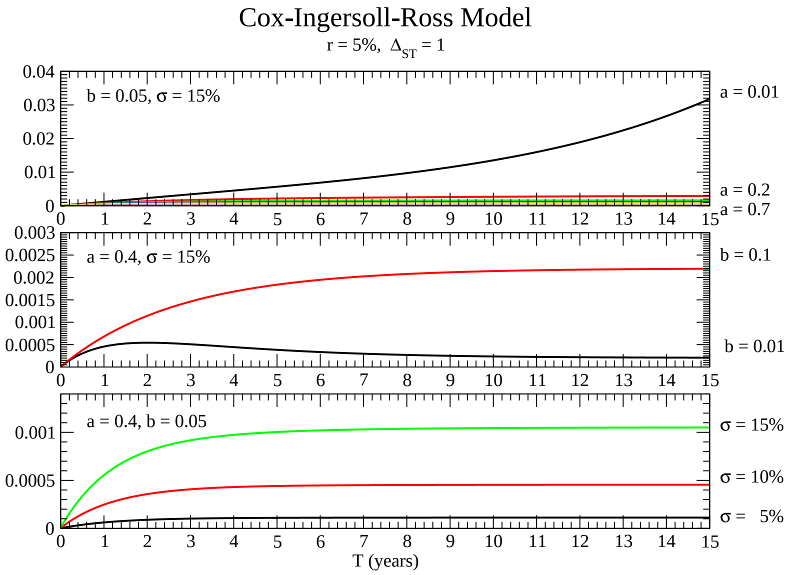

Figure 9: $CC^{LIA}(0,T,T,T+\Delta_{ST})$ as a function of the various parameters.

F.

Although F and G are coupled by equations (34), there relative contribution to the convexity corrections is a function of maturity time. Typical convexity adjustments, for various Δ_{ST}, are reported in figure 8. The first comment being that for the Cox-Ingersoll-Ross model, the corrections are smaller than for the Vasiček model. If for the later model the corrections can achieve almost 10%, for the first model they are always below 0.2%. Again, the corrections increase with the payoff time.

Figure 9 shows how the convexity adjustments change with the parameters of the Cox-Ingersoll-Ross model. In general, CC^{LIA} decreases when a takes smaller values and increases when b or σ take larger values. Figures 9 and 3 show that the functional dependence of the convexity corrections is the essentially the same for the two models investigated here. The main difference being its absolute value with $CC^{LIA} \sim 0.4\%$ for the Vasiček model and $CC^{LIA} \sim 0.1\%$ for the Cox-Ingersoll-Ross model.

For completeness, in Figure 10 we show the LIBOR in-arrears together with the LIA corrected by the convexity adjustment computed for the Cox-Ingersoll-Ross model. Note that the curves have the same structure as those computed within the Vasiček model - see Figure 4.

5 Results and Conclusions

In this paper we perform the first numerical evaluation of the unified treatment for convexity adjustments suggested in [7]. In the approach developed here, the calculation of convexity adjustments is reduced to the computation of the solutions of a system of partial differential equations subject to specific boundary conditions. The above cited

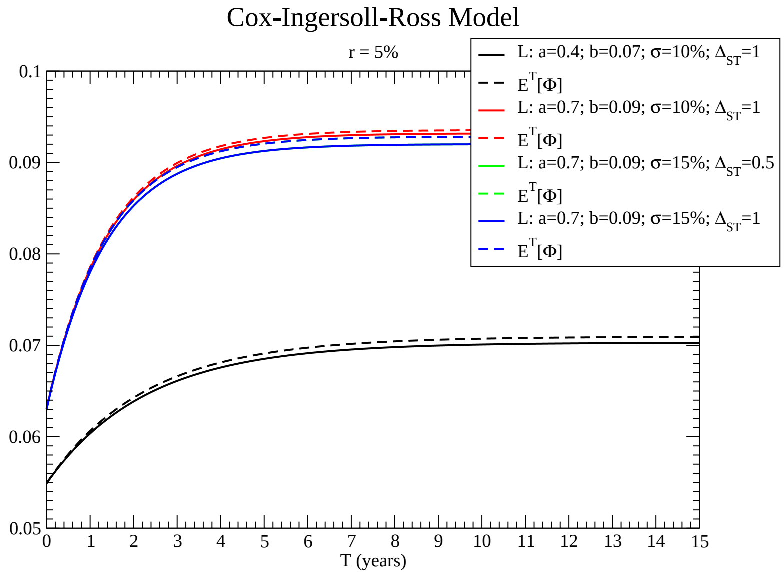

Figure 10: $L(0, T, T + \Delta_{ST})$ and $L(0, T, T + \Delta_{ST}) + CC^{LIA}(0, T, T + \Delta_{ST})$ for the Cox-Ingersoll-Ross model.

paper explores only affine term structure models. From the theoretical point of view, the work of Gaspar and Murgoci removes market inconsistencies and arbitrage opportunities present in previous calculations of the convexity adjustments.

Our numerical investigation is concerned with the computation of the convexity adjustments for LIBOR in arrears. For the evaluation of the convexity adjustments adjustments we rely on short rate models. Here we investigate the solutions of two popular models, namely the Vasiček and the Cox-Ingersoll-Ross models. Our numerical work shows that the solutions of the partial differential equations derived in [7] are smooth functions, not too difficult to handle numerically.

In general the precise value for the convexity adjustments is a function of the parameters that define the short rate model. We have found $\sim 0.4\%$ for the Vasiček model and $\sim 0.1\%$ for the Cox-Ingersoll-Ross model as typical values for the adjustments. Despite the difference in the order of magnitude for the correction, the convexity adjustments follows essentially the same type of behavior in the two short rate models explored here. Furthermore, as expected the adjustments increase with the volatility σ, with the maturity time T and with $\Delta_{ST} = S - T$. How the convexity adjustment change with the model parameterization is summarized in Figures 3 (Vasiček) and 9 (Cox-Ingersoll-Ross).

This is the first numerical investigation of the theoretical work developed by Gaspar and Murgoci in [7]. We do not explore all the types of convexity adjustments derived there, neither all the popular short rate models within the LIBOR in-arrears. The numerical techniques explored here are quite general and can be used within any short rate model and/or for the multidimensional case.

References

[1] J. C. Hull, *Options, Futures, and Other Derivative Securities.* Prentice Hall International, Inc. 6th edition, 2006.

[2] D. Pugaschevsky, " Forward cms rate adjustments", Risk, 125-128 (2001).

[3] Y. Hart, "Unifying theory", Risk, 54-55 (1997).

[4] P. S. Hagan, " Convexity conundrums: Pricing cms swaps, caps and floors".Wilmott magazine, 38-44 (2003).

[5] A. Pelsser, *Efficient Methods for Valuing Interest Rate Derivatives.* Springer Finance, Heidelberg, 2000.

[6] D. Brigo, F. Mercurio, *Interest Rate Models: Theory and Practice.* Springer Finance, Heidelberg, 2nd edition, 2006.

[7] R. M. Gaspar, A. Murgoci, "Convexity Adjustments for general Affine Term Structure Models". ADVANCE working paper series, ISEG, Technical University of Lisbon. (2008).

[8] O. Vasiček, "An Equilibrium Characterization of the Term Structure". Journal of Financial Economics **5**, 177-188 (1977).

[9] J. Cox, J. Ingersoll, S. Ross, "A Theory of the Term Structure of Interest Rates", Econometrica **53**, 385-407 (1985).

[10] A. Pelsser, "Mathematical foundation of convexity correction", Quantitative Finance 3, 59-65 (2003).

[11] T. Bjork, *Arbitrage Theory in Continuous Time.* Oxford University Press, Stockholm, 2nd edition, 2004.

[12] See, for example, W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, *Numerical Recipes. The art of scientific computing*, Cambridge University Press, 3rd edition, 2007.

[13] S. Zeytun, A. Gupta, "A Comparative Study of the Vasicek and the CIR Model of the Short Rate", Berichte des Fraunhofer ITWM, 124, pp. 1-17, (2007).