LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Mathematics II

Undergraduate Degrees in Economics and Management
Midterm Exam, April 5, 2016

Solution Topics

1. *Consider the matrix* $A = \begin{bmatrix} 2 & -6 & 2 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}$

(a) *Compute all eigenvalues of A.*

The eigenvalues of A are the solutions to the polynomial equation $|A - \lambda I| = 0$.

$$\begin{aligned} |A - \lambda I| = 0 &\Leftrightarrow (2-\lambda)((5-\lambda)^2 - 1) = 0 \\ &\Leftrightarrow \lambda = 2 \vee (5-\lambda)^2 = 1 \\ &\Leftrightarrow \lambda = 2 \vee \lambda = 4 \vee \lambda = 6 \end{aligned}$$

(b) *Determine the eigenvectors associated to the eigenvalue with smallest absolute value.*

The eigenvalue with smallest absolute value is $\lambda = 2$. The associated eigenvectors are the solutions to the linear system $Av = 2v$.

$$\begin{cases} 2v_1 - 6v_2 + 2v_3 &= 2v_1 \\ 5v_2 - v_3 &= 2v_2 \\ -v_2 + 5v_3 &= 2v_3 \end{cases} \Leftrightarrow \begin{cases} v_3 &= 3v_2 \\ v_3 &= 3v_2 \\ v_2 &= 3v_3 \end{cases} \Leftrightarrow \begin{cases} & - \\ v_3 &= 0 \\ v_2 &= 0 \end{cases}$$

All eigenvectors associated to $\lambda = 2$ satisfy $v_2 = v_3 = 0$, and $v_1 \in \mathbb{R}$. They can be written in the form

$$v = t(1, 0, 0), \quad t \in \mathbb{R}.$$

2. *Consider the quadratic form defined by* $Q(x, y, z) = 2x^2 + (1-k)y^2 + (1+k)z^2 + 2yz$. *Classify it in terms of the parameter* $k \neq 0$.

The quadratic form can be written as $Q(x, y, z) = [x\ y\ z]^T A [x\ y\ z]$, where

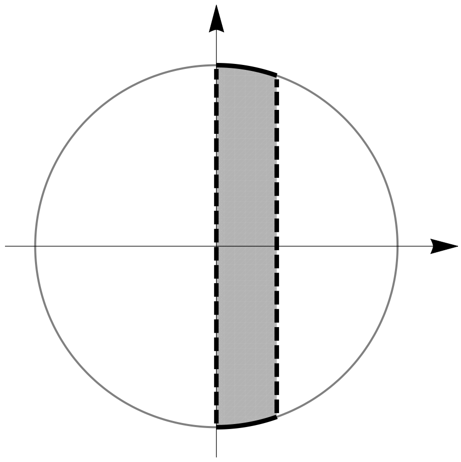
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & (1-k) & 1 \\ 0 & 1 & (1+k) \end{bmatrix}$$

The principal minors of A are given by $\Delta_1 = 2 > 0$, $\Delta_2 = 2(1-k)$ and $\Delta_3 = -2k^2 < 0$. Since $\Delta_3 \neq 0$ and the full sequence of minors does not correspond to Q being positive definite or negative definite, we can conclude that the quadratic form is indefinite, for all $k \neq 0$.

3. *Let* $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = \dfrac{\ln(1-x^2)\sqrt{9-x^2-y^2}}{\ln x}$.

(a) *Determine the domain of* f, D_f, *and represent it graphically.*

$$\begin{aligned} D_f =& \{(x,y) \in \mathbb{R}^2 : 1-x^2 > 0 \wedge 9-x^2-y^2 \geq 0 \wedge x > 0 \wedge \ln x \neq 0\} \\ =& \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \wedge x^2+y^2 \leq 3^2 \wedge x > 0 \wedge x \neq 1\} \\ =& \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \wedge x^2+y^2 \leq 3^2\} \end{aligned}$$



(b) *Determine the interior and the boundary of* D_f. *Show that* D_f *is bounded but not compact.*

$$\begin{aligned} \text{Int}(D_f) =& \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \wedge x^2+y^2 < 9\} \\ \text{Bdy}(D_f) =& \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge x^2+y^2 = 9\} \cup \\ & \{(x,y) \in \mathbb{R}^2 : x = 0 \wedge -3 \leq y \leq 3\} \cup \\ & \{(x,y) \in \mathbb{R}^2 : x = 1 \wedge x^2+y^2 \leq 9\} \end{aligned}$$

The set D_f is bounded because it is contained in a Ball, for example $D_f \subset B_4(0,0)$. However, the set is not compact because it is not closed (the dashed lines belong to the adherence of D_f but not to D_f).

4. *Consider the function* $f(x,y) = \begin{cases} \dfrac{xy^2}{x^4+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) *Discuss the continuity of* f *at* $(0,0)$.

Since $f(0,0) = 0$, the function will be continuous at $(0,0)$ if and only if $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. We need to consider two limits:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ x\neq 0}} f(x,y)$$

The first limit is zero, as f is identically zero over the set $B_1 = \{(x,y) \in \mathbb{R}^2 : x = 0\}$. It remains to show that

$$\lim_{\substack{(x,y)\to(0,0)\\ x\neq 0}} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^2} = 0.$$

But this last result is also trivial, if we note that

$$\left|\frac{xy^2}{x^4+y^2} - 0\right| = \frac{|x|y^2}{x^4+y^2} \leq \frac{|x|(y^2+x^4)}{x^4+y^2} = |x| \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

(b) *Compute* $\dfrac{\partial f}{\partial y}(0,0)$.

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

5. Let $f(x,y,z) = x^2 + xy + z\sin(xy)$. Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = z(x+y)\cos(xy) + 3x + y + \sin(xy).$$

$$\frac{\partial f}{\partial x} = 2x + y + yz\cos(xy)$$

$$\frac{\partial f}{\partial y} = x + xz\cos(xy)$$

$$\frac{\partial f}{\partial z} = \sin(xy)$$

$$\begin{aligned}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} &= (2x + y + yz\cos(xy)) + (x + xz\cos(xy)) + \sin(xy)\\ &= z(x+y)\cos(xy) + 3x + y + \sin(xy)\end{aligned}$$

Point values: **1**. (a) 1,0 (b) 1,0 **2**. 2,0 **3**. (a) 1,25 (b) 1,25 **4**. (a)1,5 (b)1,0 **5**. 1,0