Mathematics II

Undergraduate Degrees in Economics and Management

Repeat period, January 30, 2017

Part II

1. Each week an individual consumes quantities X, Y of two given goods and works for L hours, with a satisfaction level measured by the function

$$S(X,Y,L) = \frac{1}{4}\ln X + \frac{1}{4}\ln Y + \frac{1}{2}\ln(40-L), \quad (X,Y,L>0, L<40).$$

His budget is determined by the number of working hours, according to the relation $2X + 4Y = 8L$.

(a) Show that the satisfaction function $S(X,Y,L)$ does not have global extrema over the set $\Omega = \{(X,Y,L) \in \mathbb{R}^3 : X > 0, Y > 0, L > 0, L < 40\}$.

Solution: We will show that S can take arbitrarily large positive or negative values over Ω. We can do so by considering the partial function given by $f(X) = S(X,1,39) = \frac{1}{4}\ln X$, which can take any real value, since it is a continuous function of $X > 0$ such that

$$\lim_{X\to 0^+} f(x) = -\infty \quad \text{and} \quad \lim_{X\to+\infty} f(X) = +\infty.$$

(b) Assuming that there exists a triplet (X^*, Y^*, L^*) that locally maximizes $S(X,Y,L)$, given the budgetary restriction, determine it.

Solution: We are searching for local maximum points of $S(X,Y,L)$, subject to the restriction $2X + 4Y - 8L = 0$. Both the objective function and the restriction are continuously differentiable functions over the set Ω defined in (a), and the Jacobian matrix of the restrictions $J = [2, 4, -8]$ has maximal rank, and so a local maximum must occur at a critical point of the Lagrangian

$$\mathcal{L}(X,Y,L,\lambda) = \frac{1}{4}\ln X + \frac{1}{4}\ln Y + \frac{1}{2}\ln(40-L) - \lambda(2X + 4Y - 8L)$$

$$\begin{cases} \mathcal{L}'_X = 0 \\ \mathcal{L}'_Y = 0 \\ \mathcal{L}'_L = 0 \\ \mathcal{L}'_\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{4X} - 2\lambda = 0 \\ \frac{1}{4Y} - 4\lambda = 0 \\ -\frac{1}{2}\frac{1}{40-L} + 8\lambda = 0 \\ 2X + 4Y - 8L = 0 \end{cases} \Leftrightarrow \begin{cases} X = \frac{1}{8\lambda} \\ Y = \frac{1}{16\lambda} \\ L = 40 - \frac{1}{16\lambda} \\ \frac{1}{4\lambda} + \frac{1}{4\lambda} - 320 - \frac{1}{2\lambda} = 0 \end{cases} \Leftrightarrow$$

Now, the last equation yields $\lambda = 1/320$ and substituting in the previous equations we get a candidate $(X^*, Y^*, L^*) = (40, 20, 20)$. In order to show that this critical point is in fact a local maximum, we assemble the bordered Hessian matrix

$$H(X, Y, L, \lambda) = \left[\begin{array}{c|ccc} 0 & -2 & -4 & 8 \\ \hline -2 & -\frac{1}{4X^2} & 0 & 0 \\ -4 & 0 & -\frac{1}{4Y^2} & 0 \\ 8 & 0 & 0 & -\frac{1}{(40-L)^2} \end{array}\right].$$

Considering the number of variables and restrictions, the critical point may be classified using the minors $\Delta_3 = \frac{1}{200} > 0$ and $\Delta_4 = -\frac{3}{160000} < 0$, that satisfy $(-1)^1\Delta_3 < 0$ and $(-1)^4\Delta_4 > 0$. This shows that $(40, 20, 20)$ is in fact a local maximum. The optimal strategy in this situation consists in working 20 hours per week and buying 40 units of the first good and 20 units of the second, with a satisfaction level of $S(40, 20, 20) = \frac{1}{4}\ln 40 + \frac{1}{4}\ln 20 + \frac{1}{2}\ln 20 = \frac{3}{4}\ln 20 \approx 2.2468$.

Note: Since i. we were told to assume the existence of a local maximum; ii. the local maximum must occur at a critical point of the Lagrangian and iii. the Lagrangian has a single critical point; the proof using the bordered hessian was not strictly necessary.

2. Compute $\iint_\Omega (xy + 1)dx\,dy$, where Ω is the region bounded by the curves $y = x^2$ and $y = x$, for $x \in [0, 1]$.

Solution:

$$\iint_\Omega (xy+1)dxdy = \int_0^1 \int_{x^2}^{x} xy + 1dydx = \int_0^1 \left[\frac{xy^2}{2} + y\right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (\frac{x^3}{2} + x - \frac{x^5}{2} - x^2)dx = \left[\frac{x^4}{12} + \frac{x^2}{2} - \frac{x^6}{12} - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{6}$$

3. Solve the initial value problem $y'' + 2y' + y = 4e^t$, with $y(0) = 1$, $y'(0) = 2$.

Solution: Using the superposition principle, the general solution of this second order linear differential equation with constant coefficients can be written has $y(t) = y_h(t) + y_*(t)$, where $y_h(t)$ is the general solution of the associated homogeneous equation and $y_*(t)$ is a particular solution of the equation.

i. Determination of $y_h(t)$.

$$y_h'' + 2y_h' + 2y_h = 0 \Leftrightarrow (D^2 + 2D + 1)y_h = 0 \Leftrightarrow (D-1)^2 y_h = 0$$
$$\Leftrightarrow y_h(t) = (C_1 + C_2 t)e^{-t}$$

ii. Determination of $y_*(t)$.

Since the second member of the equation is e^t, we will try a particular solution of the form $y_*(t) = Ke^t$. Substituting in the differential equation we get,

$$(Ke^t)'' + 2(Ke^t)' + Ke^t = 4e^t \Leftrightarrow 4Ke^t = 4e^t \Leftrightarrow K = 1,$$

and we conclude that $y_*(t) = e^t$ is a particular solution.

iii. From i. and ii. we can write the general solution of the equation, given by

$$y(t) = y_h(t) + y_*(t) = (C_1 + C_2 t)e^{-t} + e^t$$

and we can also compute

$$y'(t) = C_2 e^{-t} - (C_1 + C_2 t)e^{-t} + e^t$$

iv. Finally, we can compute C_1, C_2 using the initial conditions

$$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases} \Leftrightarrow \begin{cases} C_1 + 1 = 1 \\ C_2 - C_1 + 1 = 2 \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ C_2 = 1 \end{cases}$$

and get the solution

$$y(t) = te^{-t} + e^t.$$

4. Solve the differential equation $x^2 y' + (x^2 - 1)y^3 = 0$, for $x \geq 1$, considering the initial condition $y(1) = 1$.

Solution: The differential equation can be rewritten as

$$x^2y' + (x^2-1)y^3 = 0 \Leftrightarrow x^2dy = -(x^2-1)y^3dx \Leftrightarrow \frac{1}{y^3}dy = \frac{1-x^2}{x^2}dx$$

and is therefore a differential equation with separable variables. The solution is implicitly defined by the equation

$$\int \frac{1}{y^3}dy = \int \left(\frac{1}{x^2} - 1\right) dx \Leftrightarrow -\frac{1}{2y^2} = -\frac{1}{x} - x - C \Leftrightarrow \frac{1}{2y^2} - \frac{1}{x} - x + C = 0.$$

Since $y(1) = 1$ the value of C can be computed from

$$\frac{1}{2\times 1^2} - \frac{1}{1} - 1 + C = 0 \Leftrightarrow C = \frac{3}{2},$$

and the solution is given implicitly by

$$\frac{1}{2y^2} - \frac{1}{x} - x - \frac{3}{2} = 0,$$

or, computing y explicitly in terms of x, by

$$y = \sqrt{\frac{x}{2x^2+3x+2}}.$$

Point values: **1**. (a) 1.0 (b) 2.5 **2**. 2.0 **3.** 2.5 **4**. 2,0

Part I

1. Classify the following statements as true or false, providing a proof or a counter-example, respectively.

(a) If $\boldsymbol{v} \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of $A \in \mathbb{R}^{n\times n}$, it cannot be associated with two different eigenvalues.

Solution: The statement is true. Let us suppose that $\boldsymbol{v} \neq 0$ is associated to two different eigenvalues of A, $\lambda_1 \neq \lambda_2$. In this case we have that

$$(A - \lambda_1)\boldsymbol{v} = 0, \quad (A - \lambda_2 I)\boldsymbol{v} = 0.$$

If we subtract the two previous equalities, we get $(A - \lambda_1 I - A + \lambda_2 I)\boldsymbol{v} = 0$, or simply $(\lambda_1 - \lambda_2)\boldsymbol{v} = 0$. Since $\lambda_1 \neq \lambda_2$ we should have $\boldsymbol{v} = 0$, which is a contradiction. Therefore we conclude that $\boldsymbol{v}$ must be associated to a single eigenvalue.

(b) If $\lambda = 2$ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ then $A - 2I = 0$.

Solution: The statement is false. For example, matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has $\lambda = 2$ as an eigenvalue, but $A - 2I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$.

2. Classify the quadratic form $Q(x, y, z) = xy + x^2 + yz + 4xz$.

Solution: The symmetric matrix associated to Q is

$$A = \begin{bmatrix} 1 & 1/2 & 2 \\ 1/2 & 0 & 1/2 \\ 2 & 1/2 & 0 \end{bmatrix}$$

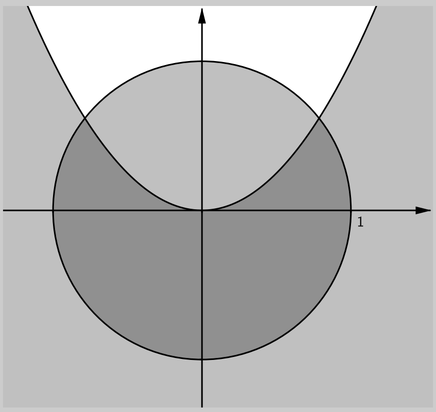
and the determinants of its principal minors are $\Delta_1 = 1 > 0$, $\Delta_2 = 1 \times 0 - 1/2 \times 1/2 = -1/4 < 0$ and $\Delta_3 = 2\times(1/2\times 1/2 - 2\times 0) - \frac{1}{2}\times(1\times 1/2 - 2\times 1/2) = 3/4 > 0$. Considering the signs of these determinants, matrix A is indefinite and so is the quadratic form Q.

3. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by the expression $f(x,y) = \sqrt{x^2 - y} + \sqrt{1 - x^2 - y^2}$.

(a) Determine the domain of f, Ω, analytically and geometrically.

Solution:

$$\begin{aligned} \Omega =& \{(x,y) \in \mathbb{R}^2 : x^2 - y \geq 0 \wedge 1 - x^2 - y^2 \geq 0\} \\ =& \{(x,y) \in \mathbb{R}^2 : y \leq x^2 \wedge x^2 + y^2 \leq 1\} \end{aligned}$$



The domain is represented by the dark grey region in the picture above.

(b) Determine the boundary of Ω and decide if the set is closed.

Solution:

$$Bdy(\Omega) = \{(x,y) \in \mathbb{R}^2 : y = x^2 \wedge x^2 + y^2 \leq 1\} \cup \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \wedge y \leq x^2\}$$

Since all boundary points are already in the set Ω, we have that $Ad(\Omega) = int(\Omega) \cup Bdy(\Omega) = \Omega$, which means that Ω is closed.

(c) Show that f has a global maximum point $(x^*, y^*) \in \Omega$ and that $1 \leq f(x^*, y^*) \leq 1 + \sqrt{2}$.

Solution: As we have seen in (b), Ω is closed and, because $\Omega \subset B_2((0,0))$, Ω is also bounded. Since Ω is closed and bounded, it is compact. Also, f is continuous, because it is the sum of two continuous functions (they are the composition of the polynomial functions $x^2 - y$ and $1 - x^2 - y^2$ with the continuous function $\sqrt{\cdot}$). Because Ω is compact and $f : \Omega \to \mathbb{R}$ is continuous, Weierstrass's theorem guarantees that f attains a global minimum and maximum over Ω. The maximum point (x^*, y^*) is a point where the global maximum value is attained.

Regarding the inequalities $1 \leq f(x^*, y^*) \leq 1 + \sqrt{2}$, we can see that **i.** Since $f(0,0) = 1$ and $f(x^*, y^*) \geq f(x,y), (x,y) \in \Omega$, we must have $f(x^*, y^*) \geq 1$; **ii.** $\sqrt{x^2 - y} + \sqrt{1 - x^2 - y^2} \leq \sqrt{1 - (-1)} + \sqrt{1 - 0^2 - 0^2} = 1 + \sqrt{2}$.

4. Show that $f(x,y,z) = \begin{cases} \dfrac{xyz}{\sqrt{x^2+y^2+z^2}} & , (x,y,z) \neq (0,0,0) \\ 0 & , (x,y,z) = (0,0,0) \end{cases}$ is continuous in $\mathbb{R}^3$.

Solution: When $(x,y,z) \neq (0,0,0)$ f is the quotient of two continuous functions: a polynomial and the square root of a continuous positive function, where the denominator does not vanish; and is therefore a continuous function. When $(x,y,z) = (0,0,0)$, f is continuous if

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x^2+y^2+z^2}} = f(0,0,0) = 0.$$

Now, since

$$\left| \frac{xyz}{\sqrt{x^2+y^2+z^2}} - 0 \right| \leq \frac{|x||y||z|}{\sqrt{x^2+y^2+z^2}} \leq \frac{(\sqrt{x^2+y^2+z^2})^3}{\sqrt{x^2+y^2+z^2}}$$
$$= (x^2+y^2+z^2) \to 0, \quad (x,y,z) \to (0,0,0)$$

we conclude that f is also continuous at $(0,0,0)$ and so it is continuous over $\mathbb{R}^2$.

5. Consider $f(x, y) = x^2 y \sin(x + y)$.

(a) Compute the partial derivatives f'_x, f'_y and show that f is differentiable in $\mathbb{R}^2$.

Solution:

$$\begin{aligned} f'_x =& 2xy \sin(x + y) + x^2 y \cos(x + y) \\ f'_y =& x^2 \sin(x + y) + x^2 y \cos(x + y) \end{aligned}$$

The partial derivatives of f only involve sums and products of continuous functions (polynomials, sines and cosines of polynomials) and so are continuous in $\mathbb{R}^2$. This is sufficient to show that f is diffeentiable in $\mathbb{R}^2$.

(b) Let $G(u, v) = f(uv, u - v)$. Using the chain rule, compute $G'_v(1, 1)$.

Solution: We start by observing that, denoting $x = uv$ and $y = u - v$, when $u = v = 1$ we have $x = 1$ and $y = 0$. The chain rule then yields,

$$\begin{aligned} G'_u =& \frac{\partial f}{\partial x}(1, 0)\frac{\partial x}{\partial u}(1, 1) + \frac{\partial f}{\partial y}(1, 0)\frac{\partial y}{\partial u}(1, 1) \\ =& 0 \times 1 + \sin 1 \times 1 = \sin 1 \end{aligned}$$

(c) Using Taylor's formula approximate f by a polynomial of degree two, when (x, y) is close to $(0, 0)$.

Solution: Using Taylor's formula we know that

$$\begin{aligned} f(x, y) \approx& f(0, 0) + x f'_x(0, 0) + y f'_y(0, 0) + \frac{1}{2!}\left(x^2 f''_{xx}(0, 0) + 2xy f''_{xy}(0, 0) + y^2 f''_{yy}(0, 0)\right) \\ =& \frac{1}{2!}\left(x^2 f''_{xx}(0, 0) + 2xy f''_{xy}(0, 0) + y^2 f''_{yy}(0, 0)\right) \end{aligned}$$

Now,

$$f''_{xx}(0, 0) = \big(2y \sin(x + y) + 2xy \cos(x + y) + 2xy \cos(x + y) - x^2 y \sin(x + y)\big)_{|x,y=0} = 0$$

$$f''_{yy}(0, 0) = \big(x^2 \cos(x + y) + x^2 \cos(x + y) - x^2 y \sin(x + y)\big)_{|x,y=0} = 0$$

$$f''_{xy}(0, 0) = ()_{|x,y=0}$$

Point values: 1. (a) 0.75 (b) 0.75 **2**. 1.5 **3.** (a) 1.0 (b) 0.75 (c) 1.0 **4**. 1.5 **5**. (a) 1.0 (b) 1.0 (c) 0.75