Local and Global Well-posedness for the critical Schrödinger-Debye System

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The Cubic Nonlinear Schrödinger Equation

$$iu_t + \Delta_x u = \epsilon |u|^2 u$$

Here,

- $u : (x,t) \in \mathbb{R}^d \times \mathbb{R} \to u(x,t) \in \mathbb{C}$;
- $\epsilon = 1$ (defocusing) or $\epsilon = -1$ (focusing).

Local Well-posedness in $H^s(\mathbb{R}^d)$

Let

$$H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}' : <\xi>^s \hat{f}(\xi) \in L^2(\mathbb{R}^d)\}, \quad <\xi> = \sqrt{1+|\xi|^2}.$$

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$$H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}' : < \xi >^s \hat{f}(\xi) \in L^2(\mathbb{R}^d)\}, \quad < \xi >= \sqrt{1 + |\xi|^2}.$$

A classical Local Well-Posedness result reads

For all initial data $u_0 \in H^s(\mathbb{R}^d)$ there exists a unique solution

$$u \in \mathcal{C}([0; T]; H^s(\mathbb{R}^d)), \quad u(0, x) = u_0,$$

where $T = T(u_0) > 0$.

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Typically,

$$T = T(\|u_0\|_s) \quad \text{is a decreasing function.}$$

Local Well-posedness in $H^s(\mathbb{R}^d)$

The heuristic scaling argument

If u is a solution, so is $u_\mu(x,t) = \mu u(\mu x, \mu^2 t)$. Also,

$$\|u_\mu(t)\|_{\dot{H}^s} = \mu^{s+1-\frac{d}{2}} \|u(\mu^2 t)\|_{\dot{H}^s}.$$

The Critical scaling is $s_c = \frac{d}{2} - 1$.

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Indeed, if $T > 0$ is the lifespan of u,

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If $s + 1 - \frac{d}{2} < 0$ (ie $s < s_c$), T_μ and $\|u_\mu(0)\|$ both decrease with μ.

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Local Well-Posedness is not expected in H^s for $s < s_c$.

Local Well-posedness in $H^s(\mathbb{R}^d)$

This issue has been adressed by many authors during the 80*s* (Ginibre, Velo, Tsutsumi among others).

The definitive result is the following:

Cazenave and Weissler, 1990

The IVP for the general Schrödinger equation

$$iu_t + \Delta_x u = \pm|u|^2 u$$

is locally well-posed in $H^s(\mathbb{R}^d)$ for $s \geq \max\{0; s_c\}$.

Global Well-posedness in the energy space $H^1(\mathbb{R}^d)$

The following quantities are formally conserved by the cNLS flow:

$$F(u) = \int |u|^2 \qquad E(u) = \frac{1}{2}\int |\nabla u|^2 + \epsilon\frac{1}{4}\int |u|^4.$$

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Global Well-Posedness: Main idea

Together with LWP, GWP is achieved by controlling $\|u(t)\|_{H^1}$
(By L^2 conservation, it is enough to control $\|\nabla u(t)\|_{L^2}$).

Global Well-posedness in the energy space $H^1(\mathbb{R}^d)$

- In the defocusing case this control is immediate:

$$E(u_0) = E(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int |u|^4 \geq \frac{1}{2}\int |\nabla u|^2.$$

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- In the focusing case, things are not that simple, since

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{4}\int |u|^4.$$

Global Well-posedness in the energy space $H^1(\mathbb{R}^d)$ - Focusing case

In dimension $d = 1$

The Gagliardo Nirenberg inequality

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{1}{4}}$$

yields

$$E(u_0) = E(u) \geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - C\|u_0\|_{L^2}^3\|\nabla u(t)\|_{L^2}$$

which implies $\|\nabla u(t)\|_{L^2} \leq C$.

Global Well-posedness in the energy space $H^1(\mathbb{R}^d)$ - Focusing case

In dimension $d = 2$

The Gagliardo Nirenberg inequality

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}$$

yields

$$E(u_0) = E(u) \geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2(1 - C\|u_0\|_{L^2}).$$

which implies $\|\nabla u(t)\|_{L^2} \leq C$ if $\|u_0\|_{L^2}$ is small enough.

Global Well-posedness in the energy space $H^1(\mathbb{R}^d)$ - Focusing case

In dimension $d = 2, 3$, the Virial inequality

$$\frac{\partial^2}{\partial t^2} \int |x|^2 |u(x,t)|^2 dx \leq 8dE(u_o)$$

implies blow-up for $E(u_0) < 0$.

Note that this can always be achieved: choosing $u_0 \neq 0$ and taking $\mu \to +\infty$,

$$E(u_{\mu 0}) = \frac{1}{2}\mu^{2-\frac{d}{2}} \int |\nabla u_o|^2 - \frac{1}{4}\mu^{4-d} \int |u_0|^4 dx \longrightarrow -\infty$$

since

$$2 - \frac{d}{2} < 4 - d \Leftrightarrow d < 4.$$

The Schrödinger-Debye System

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Note that for $\mu = 0$ this system reduces to the cNLS equation.

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$$\frac{d}{dt}E(t) = 2\lambda\mu \int (v_t)^2 dx,$$

where

$$E(t) = \int (|\nabla u|^2 + 2v|u|^2 - \lambda v^2)dx.$$

Local Well-Posedness - Previous results

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = uv \\ \mu v_t + v = \lambda |u|^2, \qquad u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases}$$

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The second equation can be solved with respect to v:

$$v(x,t) = e^{-\frac{t}{\mu}} v_0(x) + \frac{\lambda}{\mu} \int_0^t e^{-(t-t')} \mu |u(x,t')|^2 dt'.$$

Hence, the Schrödinger-Debye system can be rearranged into a single integro-differential equation:

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Local Well-Posedness - Previous results

With this approach:

B. Bidégaray (1998,2000)

Let $d = 1, 2, 3$ and $(u_0, v_0) \in H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$. Then, there exists $T > 0$ small enough such that the Schrödinger-Debye system has a unique solution

- $u \in L^\infty([0;T]; H^s(\mathbb{R}^d))$ if $s > \frac{d}{2}$;
- $u \in L^\infty([0;T]; H^1(\mathbb{R}^d))$ if $s = 1$;
- $u \in C([0;T]; L^2(\mathbb{R}^d)) \cap L^{\frac{8}{d}}([0;T]; L^4(\mathbb{R}^d))$ if $s = 0$.

Local Well-Posedness

In the framework of Bourgain spaces introduced by J. Ginibre, Y. Tsutsumi and G. Velo for the Zakharov system, we obtained the following result:

Let $d = 2, 3$ and $(u_0, v_0) \in H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$ such that

$$\max\{0, s-1\} \leq l \leq min\{2s, s+1\}.$$

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We will shortly describe this method.

Bourgain spaces

Consider the linear equation

$$iu_t = L(D)u, \qquad \widehat{L(D)u}(\xi) = p(\xi)\hat{u}(\xi).$$

(For the Schrödinger equation $iu_t + \Delta_x u = 0$, $p(\xi) = |\xi|^2$)

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Taking the Fourier transform in space and time:

$$-\tau\hat{u}(\tau,\xi) = -|\xi|^2\hat{u}(\tau,\xi) \, : \, (\tau + |\xi|^2)\hat{u}(\tau,\xi) = 0.$$

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$\hat{u}$ is supported on the paraboloid $\tau = -|\xi|^2$.

Bourgain spaces

Now, we consider a nonlinear pertubation:

$$iu_t = L(D)u + f(u).$$

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We will measure this phenomena using the norm

$$\|u\|^2_{X^{s,b}} = \| < \xi >^s < \tau + |\xi|^2 >^b \hat{u}(\tau, \xi)\|^2_{L^2(\mathbb{R}^{d+1})},$$

$$< x >= \sqrt{1 + |x|^2}.$$

Bourgain spaces

A fundamental result:

Theorem

Let $Y = Y(\mathbb{R}^d \times \mathbb{R})$ a Banach space and $b > \frac{1}{2}$.
If, for all $f \in H_x^s$ and for all $\tau_0 \in \mathbb{R}$,

$$\|e^{it\tau_0} e^{-ip(D)t} f\|_Y \leq \|f\|_{H^s}$$

then

$$X^{s,b} \hookrightarrow Y.$$

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then

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Corollary

$$\text{If } b > \tfrac{1}{2}, \qquad X^{s,b} \hookrightarrow C^0_t H^s_x.$$

Bourgain spaces

Hence, (after truncating in time), we will prove the local-posedness of

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Writing this equation in integral form yields $(S(t) = e^{-ip(D)t})$

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')f(u(t'))dt' = S(t)u_0 - iU *_t f(u).$$

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We will deal with

$$u(t) = \psi_T S(t)u_0 - i\psi_T U *_t f(u).$$

Contraction is $X^{s,b}$

$$u(t) = \psi_T S(t) u_0 - i\psi_T U *_t f(u).$$

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- Linear estimates
  - $$\|\psi_T S(t)u_0\| \leq \|\psi\|_{H_t^b}\|u_0\|_{H_x^s}.$$
  - For suitable b',

$$\|\psi_T U *_t f(u)\|_{X^{s,b}} \leq T^{1-b+b'}\|f(u)\|_{X^{s,b'}}.$$

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  - For suitable b',
    $$\|\psi_T U *_t f(u)\|_{X^{s,b}} \leq T^{1-b+b'}\|f(u)\|_{X^{s,b'}}.$$
- Nonlinear estimates
  - Estimate $\|f(u)\|_{X^{s,b'}}$ in terms of $\|u\|_{X^{s,b}}$

Back to the Schrödinger-Debye system

We write

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in integral form:

$$\begin{cases} u(t) = S(t)u_0 - i\int_0^t S(t-t')uv(t')dt' \\ \\ v(t) = e^{-\frac{t}{\mu}}v_0 + \frac{\lambda}{\mu}\int_0^t e^{\frac{-(t-t')}{\mu}}|u(t')|^2 dt' \end{cases}$$

and define Bourgain spaces adapted to u and v:

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and define Bourgain spaces adapted to u and v:

$$\|u\|_{X^{s,b}} = \| < \xi >^s < \tau + \frac{1}{2}|\xi|^2 >^b \hat{u}(\xi,\tau)\|_{L^2};$$

$$\|v\|_{H^{l,c}} = \| < \xi >^l < \tau >^c \hat{v}(\xi,\tau)\|_{L^2}.$$

Back to the Schrödinger-Debye system

As explained, one only needs to estimate the nonlinear terms

- $\|f_1(u,v)\|_{X^{s,b'}} = \|uv\|_{X^{s,b'}}$
- $\|f_1(u,v)\|_{H^{l,c'}} = \||u|^2\|_{X^{l,c'}}$

by the norms $\|u\|_{X^{s,b}}$ and $\|v\|_{H^{l,c}}$.

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We established the following (bilinear) estimates:

$$\|uv\|_{X^{s,-\frac{1}{2}+}} \leq C\|u\|_{X^{s,\frac{1}{2}+}}\|v\|_{H^{l,\frac{1}{2}+}}, \quad s \geq 0,\ l \geq \max\{0, s-1\};$$

$$\|u\overline{w}\|_{H^{l,-\frac{1}{2}+}} \leq C\|u\|_{X^{s,\frac{1}{2}+}}\|w\|_{X^{s,\frac{1}{2}+}}, \quad s \geq 0,\ l \leq \min\{2s, s-1\};$$

Global well-posedness of the D-S system

A. Corcho, J.D. Silva & FO (2011)

Let $(u_0, v_0) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R})$. Then, for all $T > 0$, there exists a unique solution

$$(u, v) \in C([0; T], H^1(\mathbb{R}^2) \times L^2(\mathbb{R})).$$

to the I.V.P. associated to the Debye-Schrödinger system.
This theorem remains valid in both focusing and defocusing cases $\lambda = \pm 1$.

Global well-posedness of the D-S system

Idea of the proof:

Compute an a priori bound for the quantity

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We write v in terms of u:

$$v(t) = e^{-\frac{t}{\mu}} v_0(x) + \frac{\lambda}{\mu} \int_0^t e^{-\frac{(t-t')}{\mu}} |u(x,t')|^2 dt'$$

Global well-posedness of the D-S system

Idea of the proof:

Compute an a priori bound for the quantity

$$f(t) = \|\nabla u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2.$$

We write v in terms of u:

$$v(t) = e^{-\frac{t}{\mu}} v_0(x) + \frac{\lambda}{\mu} \int_0^t e^{-\frac{(t-t')}{\mu}} |u(x,t')|^2 dt'$$

and $\|\nabla u\|_{L^2}^2$ in terms of the pseudo-energy $E(t)$:

$$\|\nabla u\|_{L^2}^2 = E(t) - \int (2v|u|^2 - \lambda v^2) dx.$$

Global well-posedness of the D-S system

Example of estimate:

$$\|v\|_2 \leq \|v_0\|_2 + \frac{1}{\mu}\int_0^t e^{-\frac{(t-t')}{\mu}} \|u(t')\|_{L^4}^2 dt'$$

$$\leq \|v_0\|_2 + \frac{C^2}{\mu}\int_0^t \|u(t')\|_{L^2} \|\nabla u(t')\|_{L^2} dt'$$

$$\leq \|v_0\|_2 + \frac{C^2\|u_0\|_{L^2}}{\mu}\int_0^t \|\nabla u(t')\|_{L^2} dt'$$

Global well-posedness of the D-S system

By squaring,

$$\|v\|_{L^2}^2 \leq 2\|v_0\|_2^2 + \frac{C^4\|u_0\|_{L^2}^2}{\mu^2}\left(\int_0^t \|\nabla u(t')\|_{L^2} dt'\right)^2$$

and

$$\|v\|_{L^2}^2 \leq 2\|v_0\|_2^2 + \frac{C^4\|u_0\|_{L^2}^2}{\mu^2} t \int_0^t \|\nabla u(t')\|_{L^2}^2 dt'$$

by Cauchy Schwarz.

Global well-posedness of the D-S system

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$$\|v\|_{L^2}^2 \leq 2\|v_0\|_2^2 + \frac{C^4\|u_0\|_{L^2}^2}{\mu^2}\left(\int_0^t \|\nabla u(t')\|_{L^2} dt'\right)^2$$

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$$\|v\|_{L^2}^2 \leq 2\|v_0\|_2^2 + \frac{C^4\|u_0\|_{L^2}^2}{\mu^2} t \int_0^t \|\nabla u(t')\|_{L^2}^2 dt'$$

by Cauchy Schwarz.
Finally,

$$\|v\|_{L^2}^2 \leq 2\|v_0\|_2^2 + \frac{C^4\|u_0\|_{L^2}^2}{\mu^2} t \int_0^t f(t') dt'.$$

Global well-posedness of the D-S system

One can show that

For all $t \leq T_\mu = T_\mu(\|u_0\|_2)$,

$$f(t) \leq \alpha_0 + \alpha_1 \int_0^t f(t')dt',$$

where α_0 and α_1 depend exclusively on the initial data.