Dispersive equations and propagation of solitons

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1. Dispersive equations
2. Physical Framework
3. A few recent results

Linear dispersive equations

Let us consider a plane wave

$$u(\vec{x}, t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

propagating in the $\vec{k}$ direction with phase velocity $v_p = \dfrac{\omega}{k}$, where $k = \|\vec{k}\|$.

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Different frequencies travel at the different speeds: the wave will disperse.

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- The heat equation: $u_t - c^2\Delta u = 0$.
  No dispersion! $(-i\omega + c^2k^2 = 0)$.

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Indeed, $\lim_{t \to +\infty} \|u(\,.\,, t)\|_\infty = 0$, but, multiplying the equation by $\overline{u}$ and integrating the imaginary part,

$$\frac{d}{dt} \int_{\mathbb{R}} |u(x,t)|^2 dx = -Im \left(\int |\nabla u(x)|^2 dx \right) = 0,$$

hence, for all $t \geq 0$,

$$\|u\|_{L^2} = \|u_o\|_{L^2}.$$

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Essentiallly 3 outcomes: blow-up, global existence, solitons.

solitons

First observed by John Scott Russel in 1834, in the

"(...)assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed."

"(...) after a chase of one or two miles I lost it in the windings of the channel."

Physical framework

Nonlinear dispersive equation arise in the study of wave propagation in dispersive media.

- Magneto-HydroDynamics

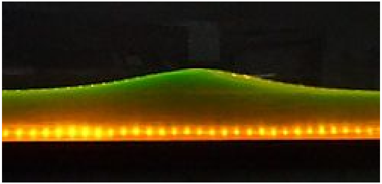
$$\begin{cases} \partial_t \rho_M + \nabla.(\rho_M \mathbf{u}) = 0 \\ \rho_M(\partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u}) = -\frac{\beta}{\gamma}\nabla(\rho_M^\gamma) + (\nabla \times \mathbf{b}) \times \mathbf{b} \\ \partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \frac{1}{R_i}\nabla \times (\frac{1}{\rho_M}(\nabla \times \mathbf{b}) \times \mathbf{b}) \\ \nabla.\mathbf{b} = 0, \end{cases}$$

  Propagation of waves (Langmuir waves, Alfvén waves) in magnetized plasmas.
- Water waves (Navier-Stokes equations)
  Propagation of waves in shallow waters.

Main problems

- Well-posedness.
  The models studied are asymptotic approximations!
  (Ex:
  $$u_t + \epsilon^2 \Delta u + \epsilon^{10} \Delta^2 u = 0.)$$

- Existence and stability of solitary waves.

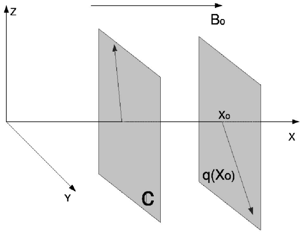


A few recent results

- FO - *Stability of the solitons for the one-dimensional Zakharov-Rubenchik equation*, Physica D (2003).

$$\begin{cases} i\partial_T q + \omega\partial_{XX} q - k(u - \frac{v}{2}\rho + \alpha|q|^2)q = 0 & \text{(a)} \\ \epsilon\partial_T \rho + \partial_X(u - v\rho) = -k\partial_X|q|^2 & \text{(b)} \\ \epsilon\partial_T u + \partial_X(\beta\rho - vu) = \frac{k}{2}v\partial_X|q|^2 & \text{(c)}, \end{cases}$$

B is the transverse magnetic field, u is the ion speed in the (Ox) direction and ρ the density of mass.



A few recent results

- Global well-posedness for initial data in $H^2 \times H^2 \times H^1$.
- Existence and stability of solitary waves of the form

$$q(x,t) = e^{i\omega t} A(x-ct), \;\; \rho(x,t) = B(x-ct), u(x,t) = C(x-ct).$$

- FO - *A class of non-local operators for Vorticity waves*, Applicable Analysis (2005)

Study of the dispersion of equations of the form

$$A_t + L(A) = 0, \quad \text{where } \widehat{L(A)}\xi = i\xi^3 \log(|\xi|).$$

-

$$|A(t,x)| \le \frac{C}{|t|^3}.$$

A few recent results

- João Paulo Dias, Mário Figueira and FO
  Existence of local strong solutions for a quasilinear Benney Equation, Comptes Rendus de l' A. S. Paris -(2007).

- Local well-posedness of the Benney-like system (interaction of short and long waves)

$$iu_t + u_{xx} = |u|^2 u + uv,$$

$$v_t + (f(v))_x = |u|^2_x$$

in $H^3 \times H^2$.

A few recent results

- S. Antontsev, João Paulo Dias, Mário Figueira and FO, *Non-existence of global solutions for a quasilinear Benney system*, Journal of Mathematical Fluid Mechanics (2009)

- Non existence of global solutions in the half-plane for the previous Benney-like system

$$iu_t + u_{xx} = |u|^2 u + uv,$$

$$v_t + (f(v))_x = |u|^2_x$$

A few recent results

- J. Silva, M. Panthee and FO, *On the Cauchy problem for the Zakharov-Schulman systems*, (2009)

-Local and global well-posedness of the system

$$iu_t + \mathcal{L}_1 u = |u|^2 u + uv,$$
$$\mathcal{L}_2 v = \mathcal{L}_3(|u|^2).$$

- J.P. Dias, M. Figueira FO, *Well-posedness and existence of bound-states for a coupled Schrödinger gKdV system*, Nonlinear Analysis (2010).

- Global well-posedness and existence of solitary waves for the system

$$iu_t + u_{xx} = \alpha uv + \beta |u|^q u,$$
$$v_t + v_{xxx} + a(v)v_x = \gamma(|u|^2)_x.$$

Thank you for your attention