On a Zakharov-type Equation for Alfvén waves

Filipe Oliveira

Centro de Matemática e Aplicações - Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa

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- The MHD equations read:

$$\begin{cases} \partial_t \rho_M + \nabla.(\rho_M \mathbf{u}) = 0 \\ \rho_M(\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u}) = -\frac{\beta}{\gamma}\nabla(\rho_M^\gamma) + (\nabla \times \mathbf{b}) \times \mathbf{b} \\ \partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \frac{1}{R_i}\nabla \times (\frac{1}{\rho_M}(\nabla \times \mathbf{b}) \times \mathbf{b}) \\ \nabla.\mathbf{b} = 0, \end{cases}$$

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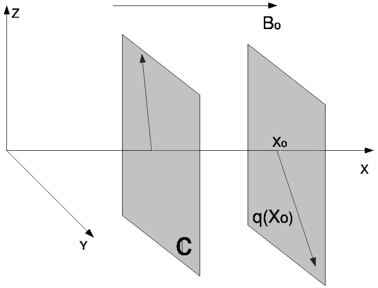
where **b** is the magnetic field, ρ the density of mass and **u** the fluid speed.

We present here a uni-dimensional asymptotic model for the evolution of wave trains of Alfvén waves with wave number k and frequency $\widetilde{\omega}$, in a frame travelling at the Alfvén-wave group velocity $v = 2\widetilde{\omega}^3 k^{-1}(k^2 + \widetilde{\omega}^2)^{-1}$.



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\begin{cases}
i\partial_T B + \omega \partial_{XX} B - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 & \text{(a)} \\
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(X, T) has been scaled: $X = \epsilon(x - vt)$ and $T = \epsilon^2 t$.

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B is the transverse magnetic field, u is the ion speed in the (Ox) direction and ρ the density of mass.

We obtain here the Zakharov-Rubenchik equation, introduced as an universal model for the interaction of long and short waves (1972).

Well posedness

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Theorem 1

The Zakharov-Rubenchik system is globally well-posed in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

First, a change of variables:

$$\begin{cases} iB_t + B_{xx} + \psi_1 B + \psi_2 B + |B|^2 B = 0 \\ \psi_{1tt} - \psi_{1xx} = |B|^2_{xx} \\ \psi_{2t} - \psi_{2x} = |B|^2_x \end{cases} \tag{1}$$

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The difficulty here is the derivative loss in the nonlinear terms.

The system can be re-written without derivative-loss:$[F \to B_t]$:

$$\begin{cases} iF_t + F_{xx} + (\psi_1 + \psi_2 + B)F \\ \qquad\qquad + (\psi_{1t} + \psi_{2t} + \overline{F}\tilde{B})\tilde{B} = 0 \\ \psi_{1tt} - \psi_{1xx} = |B|^2_{xx} \\ \psi_{2t} - \psi_{2x} = |B|^2_x, \end{cases}$$

where

$$\begin{aligned} \tilde{B}(x,t) &= B_o(x) + \int_0^t F(x,s)ds \\ B(x,t) &= (\Delta - 1)^{-1}\mathcal{A}(F, \psi_1, \psi_2, \tilde{B}). \end{aligned}$$

Using Strichartz-type estimates for the free Schrödinger group, we can now obtain the existence of local (strong) solutions via a fixed-point in the Banach space

$$\begin{aligned}\|(F,\psi_1,\psi_2)\|_{X(T)} &= \|F\|_{L^\infty(0,T,L^2)} + \|F\|_{L^6(0,T,L^6)} \\ &+ \|\psi_1\|_{L^\infty(0,T,H^1)} + \|\psi_2\|_{L^\infty(0,T,H^1)} \\ &+ \|\psi_{1t}\|_{L^\infty(0,T,L^2)} + \|\psi_{2t}\|_{L^\infty(0,T,L^2)}.\end{aligned}$$

To obtain global solutions, we need to compute some invariants:

The following quantities are conserved by the Zakharov-Rubenchik flow:

$$
\begin{aligned}
I_1(t) &= \int_{\mathbb{R}} |B|^2 \\
I_2(t) &= \frac{\omega}{2}\int_{\mathbb{R}} |B_x|^2 + \frac{kq}{4}\int_{\mathbb{R}} |B|^4 + \frac{k}{2}\int_{\mathbb{R}} (u - \frac{v}{2}\rho)|B|^2 \\
&\quad + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 - \frac{v}{2}\int_{\mathbb{R}} u\rho, \\
I_3(t) &= \epsilon\int_{\mathbb{R}} u\rho + \frac{i}{2}\int_{\mathbb{R}} (B\overline{B_x} - B_x\overline{B}).
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Using these quantities, One can show the a priori estimation

$$\forall t \leq T, \ \|(F, \psi_1, \psi_2)\|_{X(T)} \leq D(T),$$

where D is a continuous function. This is enough to prove that the solutions ares global (absence of blow-up)

Existence of solitary-wave solutions

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Inserting this in the ZR equation, we find that $Q_c(t)$ is a solution iff $R(x) = e^{\frac{-icx}{2\omega}}A(x)$ satisfies

$$R'' - ER - (\frac{k}{\omega}a - \frac{v}{2}b + q)R^2R = 0,$$

where $E = \frac{1}{w}(\lambda - \frac{c^2}{4\omega})$ and

$$a = a(c) = \frac{k(-\beta + \frac{v}{2}(c\epsilon + v))}{\beta - (c\epsilon + v)^2}, \quad b = b(c) = \frac{k(-c\epsilon - \frac{v}{2})}{\beta - (c\epsilon + v)^2}.$$

For $E > 0$, it is known that

$$R'' - ER - (\frac{k}{\omega}a - \frac{v}{2}b + q)R^2 R = 0$$

possesses a unique positive exponential decreasing solution, provided that $a - \frac{v}{2}b + q < 0$.
This last condition holds provided that ϵ is small enough. Hence:

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This last condition holds provided that ϵ is small enough. Hence:

Existence of ground states

For ϵ small enough, the Zakharov-Rubenchik system possesses soliton solutions.

Orbital Stability of ground states

We Introduce the orbit $Q = (B, u, \rho) \in H^1 \times L^2 \times L^2$:

$$\mathcal{O}(Q) = \{e^{i\alpha}B(. + x_o), u(. + x_o), \rho(. + x_o)/\alpha, x_o \in \mathbb{R}\}.$$

We set the distance between the orbit $\mathcal{O}(Q)$ and the orbit of a ground state $\mathcal{O}(Q_R)$ at a time t:

$$d(t) := d(Q(t), Q_R(t)) := \inf \alpha, x_o\{\|Q_{\alpha,x_o}(t) - Q_R(t)\|\}.$$

By building an appropriate Lyapunov invariant, we can prove the following:

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Orbital stability

For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|Q_o - Q_R(0)\| < \delta \Rightarrow \forall t \geq 0 \ , \ d(t) < \epsilon.$$

The adiabatic limit

In the adiabatic limit ($\epsilon \to 0$):

$$i\partial_T B + \omega \partial_{XX} B - k(u - \tfrac{v}{2}\rho + q|B|^2)B = 0 \ \text{(a)}$$
$$\epsilon\partial_T \rho + \partial_X(u - v\rho) = -k\partial_X|B|^2 \ \text{(b)}$$
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u and ρ become slaved to the magnetic field amplitude:

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$$B^{(\epsilon)} \to B \ ? \quad \text{If so, in what sense?}$$

Main Theorem

Assume $\tilde{\omega} < 0$, $\beta - v^2 > 0$ and $v < 0$.
Let $s > \frac{3}{2}$, $\epsilon < 1$ and

$$(B_o, \rho_o, u_o) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}).$$

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Then there exists $T_o > 0$ independent of ϵ such that the Zakharov-Rubenchik possesses a unique solution

$$(B^{(\epsilon)}, \rho^{(\epsilon)}, u^{(\epsilon)}) \in \mathcal{C}^o([0; T_o]; H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R})).$$

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Furthermore, if $u_o - v\rho_o = -k|B_o|^2$ and $\beta\rho_o - vu_o = k\frac{v}{2}|B_o|^2$,

$$B^{(\epsilon)} \longrightarrow B \text{ in } C^o([0; T]; C^2_{loc})$$

where B is the solution to the NLS equation (15) for initial data $B(0, x) = B_o(x)$.

Key: if $\tilde{\omega} < 0$, $\beta - v^2 > 0$ and $v < 0$,putting

$$(V, F, G) := (\epsilon\partial_x^{-1}(u + \frac{v}{2}\rho)_t, u - v\rho + k|B|^2, \beta\rho - vu + k\frac{v}{2}|B|^2)$$

and

$$(\alpha, \beta, \gamma, \delta) := \sqrt{2}(Re(B), Im(B), Re(B_x), Im(B_x)),$$

$$Y = (V, F, G, \alpha, \beta, \gamma, \delta)$$

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satisfies the perturbed symmetric hyperbolic system:

$$Y_t + \left(\frac{1}{\epsilon}M + N(Y)\right) Y_x + R(Y) + AY_{xx} = 0.$$

Here, M, $N(Y)$ are symmetric matrixes, A is antisymmetric and $R(Y)$ is a nonlinear term.

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Here, we will extend it to the case where the "perturbation"

$$AY_{xx}$$

is present.

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We prove the following Lemma:

Lemma

Let $s > \frac{3}{2}$ and $W(x,t) \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R})^7)$. There exists a one parameter semigroup $\{U(t)\}_{t\geq 0}$ acting on $H^s(\mathbb{R})^7$:

$$\begin{array}{rccc} U(t): & H^s(\mathbb{R})^7 & \rightarrow & H^s(\mathbb{R})^7 \\ & Y_o & \rightarrow & U(t)Y_o \end{array}$$

which generates the solution $Y(x,t) = U(t)Y_o \in C(\mathbb{R}_+, H^s(\mathbb{R})^7) \cap C^1(\mathbb{R}_+, H^{s-2}(\mathbb{R})^7)$ of the I.V.P. (2) for initial data Y_o.

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Lemma

Let $s > \frac{3}{2}$ and $W(x,t) \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R})^7)$. There exists a one parameter semigroup $\{U(t)\}_{t\geq 0}$ acting on $H^s(\mathbb{R})^7$:

$$\begin{array}{rccc} U(t): & H^s(\mathbb{R})^7 & \rightarrow & H^s(\mathbb{R})^7 \\ & Y_o & \rightarrow & U(t)Y_o \end{array}$$

which generates the solution
$Y(x,t) = U(t)Y_o \in C(\mathbb{R}_+, H^s(\mathbb{R})^7) \cap C^1(\mathbb{R}_+, H^{s-2}(\mathbb{R})^7)$ of the I.V.P. (2) for initial data Y_o.
Moreover, for $T > 0$ and for every $f \in H^s(\mathbb{R})^7$,

$$\|U(t)f\|_s \leq e^{CT \sup_{0\leq t\leq T} \|W(t)\|_s^2} \|f\|_s, \quad t \in [0; T]. \tag{3}$$

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Hence

$$\frac{d}{dt}\|Y(t)\|_s^2 = -\int < \Lambda^s(N(W(t))Y_x), \Lambda^s Y >\leq C(t)\|Y(t)\|_s^2$$

$$\text{where } C(t) = C\|W(t)\|_s^2.$$

End of the proof of the lemma

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We prove the existence of a fix-point for the application

$$\begin{array}{rcll} \Psi & : & E(T) & \rightarrow & C([0;T], H^s(\mathbb{R}^7)) \\ & & W & \rightarrow & Y = U(t)Y_o + \int_0^t U(t-\tau)R(W(\tau))d\tau, \end{array}$$

where $E(T) = \{W \in X\ ;\ \|W\|_X \leq 2K\}$, is a closed convex subset of $X(T) = C([0;T]; H^s(\mathbb{R}^7))$ and K is such that $\|Y_o\| \leq K$. (Ψ is a contraction for T small enough).

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We obtain a solution such that

$$\sup_{0\leq t\leq T} \|Y(t)\|_s \leq 2K(\text{ we can prove that } \sup_{0\leq t\leq T} \|Y_t(t)\|_{s-2} \leq C)$$

End of the proof of the main theorem:

$$Y = (V^{(\epsilon)}, F^{(\epsilon)}, G^{(\epsilon)}, \alpha^{(\epsilon)}, \beta^{(\epsilon)}, \gamma^{(\epsilon)}, \delta^{(\epsilon)}),$$

$$F^{(\epsilon)} = u^{(\epsilon)} - v\rho^{(\epsilon)} + k|B^{(\epsilon)}|^2 \text{ and } G^{(\epsilon)} = \beta\rho^{(\epsilon)} - vu^{(\epsilon)} - k\frac{v}{2}|B^{(\epsilon)}|^2.$$

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$$\|G^{(\epsilon)}\|_\infty \leq C\|D^{s-1}G^{(\epsilon)}\|_o^{\frac{1}{2(s-1)}} \|G^{(\epsilon)}\|_o^{1-\frac{1}{2(s-1)}} \leq C\epsilon^{\frac{1}{2(s-1)}}.$$

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We obtain the same kind of estimates for $F^{(\epsilon)}$:

$$F^\epsilon, G^\epsilon \to 0 \text{ in } \mathcal{C}([0;T]\times\mathbb{R}) \quad \text{(Strong topology)}.$$

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But $u_t^{(\epsilon)}$ is also bounded in $C^o([0;T];H^{s-2}(\mathbb{R})$:

By a standard argument
(Arzela Ascoli Theorem in time+Rellich compactness Theorem in space+interpolation),

$$u^{(\epsilon)} \to u \text{ in } \mathcal{C}^o([0;T], H^{s-\epsilon}_{loc})\text{(strong topogy)}.$$

Taking the limit and putting $B^{(\epsilon)} = \alpha^{(\epsilon)} + i\beta^{(\epsilon)}$:

$$B^{(\epsilon)} \to B \text{ in } \mathcal{C}^o([0; T], H^{s+1-\epsilon}_{loc}),$$

where B satisfies

$$i\partial_T B + \omega B_{XX} + \frac{kv}{4(\beta - v^2)}|B|^2 B = 0.$$

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THE END

Thank you for your attention.