NONTRIVIAL GROUND STATES FOR COOPERATIVE CUBIC SCHRÖDINGER SYSTEMS

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We will consider the following coupling of Schrödinger systems:

$$\begin{cases} i\partial_t \phi_j - \Delta\phi_j = \mu_j |\phi_j|^{2q-2}\phi_j + \phi_j|\phi_j|^{q-2}\sum_{k\neq j}\beta_{j,k}|\phi_k|^q \\ \phi_j \in H_0^1(\Omega,\mathbb{C}),\ j = 1,\ldots,n \end{cases}$$

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$n = 2$ equations:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{2q-2} u + \beta u |u|^{q-2} |v|^q & \text{in } \Omega \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{2q-2} v + \beta v |v|^{q-2} |u|^q & \text{in } \Omega \\ u_1, u_2 \in H_0^1(\Omega) \end{cases}$$

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There is an associated energy:

$$E(u,v) = \int \left(\lambda_1 |u|^2 + \lambda_2 |v|^2 + |\nabla u|^2 + |\nabla v|^2 \right)$$

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Otherwise, they are said **Semitrivial**.

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Theorem (FO, 2015)

- If $1 < q < 2$, then for all $b > 0$ the system admits Fully Nontrivial Ground States;
- If $q \geq 2$ and $\beta \geq \frac{2^q - 1}{2} \omega^{1+\frac{q}{2}} - \frac{1}{2} \omega^{-\frac{q}{2}}$, where $\omega = \lambda_2/\lambda_1$, then the system admits Fully Nontrivial Ground States.

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Referee Report: The author is invited to have a look at the following publication containing all results which the author proved in his submission: Mandel, R. : Minimal energy solutions for cooperative nonlinear Schrdinger systems, Nonlinear Differential Equations and Applications NoDEA, http://dx.doi.org/10.1007/s00030-014-0281-2.

The general system

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Theorem
(Hugo Tavares, F.O., Advanced Nonlinear Studies, 2016)
Let $\lambda_i, \mu_i, \beta_{j,k} > 0$. Then, for $1 < q < 2$, the system admits a Fully Nontrivial Ground State.

For $q = 2$ and $k \geq 3$ equations , the situation is more complex.

$$-\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + \beta u_i \sum_{j \neq i} u_j^2 \text{ in } \mathbb{R}^N$$

First Existence Results of Nontrivial Groundstates

Theorem (Z. Liu, Z.-Q. Wang, 2010)

$$\lambda_1 = \ldots = \lambda_k \qquad \beta > k(k-1)\max\{\mu_i\} - \frac{d-1}{d}\sum_j \mu_j$$

Then all groundstates are nontrivial.

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Later improved to:

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Nontrivial groundstates for different λ_i's

Theorem (Correia, Tavares, F.O., 2015)

Let $k \geq 3$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then there exists

$$\alpha = \alpha(\lambda_1/\lambda_2, n, N) < 1$$

such that, if

$$\lambda_n \leq \alpha\lambda_2,$$

then there exists a constant $B = B(\lambda_i, \mu_i) > 0$, such that, for $\beta > B$, all ground states are nontrivial.

Idea: λ_1 and λ_2 can be arbitrarily far, but all other λ_i need to be close to $\max\{\lambda_1, \lambda_2\}$.

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Is this result optimal?

In a qualitative way, yes!

Theorem (Correia, Tavares, FO 2015)

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There exists a constant $\Lambda = \Lambda(\lambda_1/\lambda_2)$ *such that, if*

$$\lambda_2 \Lambda \leq \lambda_i \text{ for some } i \geq 3, \text{ and } \beta > \max\{\mu_1, \ldots, \mu_k\},$$

then every groundstate solution is semitrivial.

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Dear authors,
Your manuscript *Semitrivial vs. fully nontrivial ground states in cooperative cubic Schrödinger systems with* $d \geq 3$ *equations* has been evaluated by two referees. Both evaluators have been very positive and I am glad to announce that your manuscript is accepted.(...)
Cedric Villani
Editor, Journal of Functional Analysis