Lisbon School of Economics & Management
Universidade de Lisboa

MATHEMATICS ADMISSION EXAM

International students

Date: 16/02/2024 **Duration:** 180 min

Part 1

Select the correct option for each of the following questions. You do not need to present your calculations. Each correct answer is awarded 1 point. Each incorrect answer is penalized in 0,2 points.

1. Consider the natural numbers with 5 digits. Among these numbers, how many have exactly three digits equal to 8 and are less than 40000?

(A) 108 (B) 120 (C) 300 (D) 128

2. There is a row in Pascal's Triangle where the eighth and twelfth elements are equal. What is the largest element of the next row?

(A) 28600 (B) 184756 (C) 92378 (D) 75582

3. Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$, defined by the expression $f(x) = \ln(e^2x^3) - \ln x$. Which of the following expressions also defines f?

(A) $2(1 + \ln x)$ (B) $2 - \ln x$ (C) $3 + \ln x$ (D) $2 + 3\ln x$

4. Consider the complex number $z = a+ia,\ a \in \mathbb{R}\backslash\{0\}$. Which of the following statements is **true**?

(A) $2z$ is a real number.
(C) z^2 is a real number.
(B) z^2 is a pure imaginary number.
(D) $|z| = a$.

5. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by the expression $f(x) = e^{x^2+x}$, and let M be the intersection point between the graph of f and the y-axis. Which of the following equations represents the tangent line to the graph of f at point M?

(A) $y = x + 1$ (B) $y = x + e$ (C) $y = ex - e$ (D) $y = ex + 1$

6. Consider a function f with domain $\mathbb{R}$ and image $[-3, 5]$. What is the image of function g, defined in $\mathbb{R}$ by the expression $g(x) = -f(x + 3) + 3$?

(A) $[3, 11]$ (B) $[1, 9]$ (C) $[-3, 5]$ (D) $[-2, 6]$

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by the expression $f(x) = \ln(1 + \sin^2 x)$. Which of the following expressions defines f', the derivative of f?

(A) $\dfrac{1}{1 + \sin^2 x}$ (B) $\dfrac{2\cos x}{1 + \sin^2 x}$ (C) $\dfrac{2\sin x \cos x}{1 + \sin^2 x}$ (D) $\dfrac{2\sin x}{1 + \sin^2 x}$

Part 2

Provide a detailed justification for each one of your answers.

1. Consider the complex number $\omega = 2e^{i\frac{\pi}{3}}$.

(a) Solve in $\mathbb{C}$ the equation $z^3 + \overline{\omega} = 0$.

(b) Determine the smallest value of $n \in \mathbb{N}$ such that ω^n is a negative real number.

2. Let Ω be the sample space associated with a certain random experiment. Given an event $X \subset \Omega$, $P(X)$ denotes the probability of X, and $\bar{X}$ denotes the complement of X.

Given events $A \subset \Omega$ e $B \subset \Omega$ such that

$$P(\overline{\overline{A} \cap \overline{B}}) = 3P(A \cap B) \text{ and } P(A \cap B) = P(A \cap \overline{B}),$$

show that

$$P(\overline{A} \cup \overline{B}) = P(A \cup \overline{B}).$$

3. A professional basketball player has a probability $p = 0.9$ of success in a free throw. This player is going to attempt 5 consecutive free throws.

(a) What is the probability of successfully making all 5 shots?

(b) What is the probability of failing exactly 2 throws?

4. Consider the real function defined by the expression

$$f(x) = 4x + \sin(2x).$$

Using exclusively analytic methods:

(a) Study f regarding its monotony and local extrema.

(b) Determine the domain and the image of the derivative of f, f'.

(c) Determine the values of x for which $f''(x) = 0$.

(d) Investigate the existence of asymptotes to the graph of f.

5. Let f be a function with domain $\mathbb{R}$ defined by:

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{e^{x-1} - 1} & \text{se } x < 1 \\ \\ \ln(ex) & \text{se } x \geq 1 \end{cases}$$

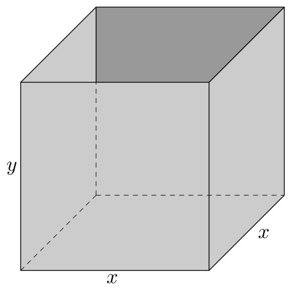
(a) Study the continuity of f in $\mathbb{R}$.

(b) Without performing any calculations, justify that f has a global maximum and a global minimum in $[2, 3]$.

(c) Compute $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

(d) Using the intermediate value theorem, show that there is at least one $c \in]2, 3[$ such that $f(c) = 2$.

6. The task at hand is to create a box with a square-shaped base that measures x cm, has a height of y cm, and remains open at the top (see figure). The box must have a volume of $32\,\text{cm}^3$.



(a) Show that the surface area of the box is given by

$$A(x) = \frac{128}{x} + x^2.$$

(b) Determine the dimensions of the box so that the surface area is as small as possible. Indicate the minimum value of the surface area.

Scores:

Part I

Question	1	2	3	4	5	6	7
Score	1	1	1	1	1	1	1

Part 2

Question	1a	1b	2	3a	3b	4a	4b	4c	4d	5a	5b	5c	5d	6a	6b
Score	1	1	1	1	1	0.5	0.5	1	1	1	0,5	0,5	1	1	1