ISEG Lisbon School of Economics & Management
Universidade de Lisboa

Mathematics admission exam

International students

Solution Topics

Date: 06/06/2023 **Duration:** 180 min

Part 1

1. The access password to a certain internet service is made of six characters. Two of those characters must be digits (0, 1, 2, 3, 4, 5, 6, 7, 8 or 9) and the remaining four characters must be uppercase vowels (A, E, I, O or U). For instance, AA2E2U and U1IEA8 are valid passwords. How many valid passwords can be formed in this way?

(A) 937500 (B) 318750 (C) 1875000 (D) 468750

2. The sum of all elements in a certain line in Pascal's triangle is 4096. What is the fourth element in that line?

(A) 286 (B) 495 (C) 220 (D) 715

3. Consider a function defined in $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ through the expression $f(x) = 2\log_5\left(\frac{x^3}{25}\right)$. Which of the following expressions can also be used to define f?

(A) $3\log_5(x)$ (B) $6\log_5(x) - 4$ (C) $6\log_5\left(\frac{x}{25}\right)$ (D) $6\log_5(x) - 50$

4. Consider a function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \sin(e^{3x})$. Which of the following expressions defines the derivative of f?

(A) $\cos(3e^{3x})$ (B) $3e^{3x}\cos(3e^{3x})$ (C) $3\cos(e^{3x})$ (D) $3e^{3x}\cos(e^{3x})$

5. Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$, defined by $f(x) = \ln(3x)$, as well as a point M, corresponding to the intersection of the graph of f with the horizontal axis. Which of the following equations defines the tangent to the graph of f at point M?

(A) $y = x - \frac{1}{3}$ (B) $y = 6x - 2$ (C) $y = 6x - 3$ (D) $y = 3x - 1$

6. How many complex solutions of the equation $z^9 = 2$ have their geometric images in the second quadrant?

(A) None (B) Two (C) Three (D) Four

7. Consider a function f with domain $\mathbb{R}$ and image $[-3, 5]$. What is the image of function g, defined in $\mathbb{R}$ through the expression $g(x) = |f(x-1)| + 3$?

(A) $[6, 8]$ (B) $[5, 7]$ (C) $[3, 5]$ (D) $\mathbf{[3, 8]}$

Part 2

1. Consider a complex number, z, such that $\text{Re}(z) > 0$ and $\text{Im}(z) = -\text{Re}(z)$. To which quadrant in the complex plane belongs the geometrical image of z^{27}?

2. Let Ω be the space of events associated to a certain random experiment. Given an event $X \subset \Omega$, denote by $P(X)$ the probability of X and by $\bar{X}$ the complementary event.

Consider two events A, B such that $P(B) < 1$. Show that $P(\bar{B}) \neq 0$ and that

$$P(A \cap B) = P(A) - P(\bar{B}) + P(\bar{A}|\bar{B}) \times P(\bar{B}).$$

3. Suppose that five 1 euro coins and six fifty cents coins are randomly placed in a 4×4 checker board.

(a) What is the probability of one of the four horizontal lines being left with no coins?

(b) What is the probability of one of the diagonals being totally filled with coins having the same face value?

4. Consider function f, with domain $\mathbb{R}^+$, defined by the expression

$$f(x) = 3x - 2\ln x + \frac{1}{x}.$$

Using exclusively analytical methods:

(a) Study the behaviour of f regarding monotony and the existence and nature of local minima/maxima.

(b) Study function f regarding the regions where its graph is up or down concave.

(c) Study the existence of asymptotes to the graph of f.

(d) Draw the graph of f.

5. Let f be the function with domain $\mathbb{R} \setminus 0$ defined by

$$\begin{cases} \dfrac{7x}{e^{3x} - 1}, & \text{if } x > 0 \\ \dfrac{\sin(7x)}{3x}, & \text{if } x < 0 \end{cases}.$$

Answer the following questions using exclusively analytical methods.

(a) What is the value that must be assigned to $f(0)$ in order to extend f to $\mathbb{R}$ as a continuous function? Justify.

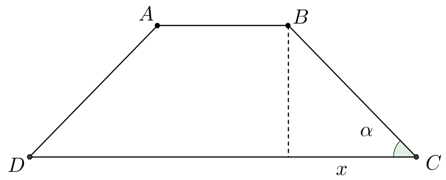
(b) Show, using the intermediate value theorem, that there exists $x_0 \in \left]-\pi, -\frac{\pi}{14}\right[$ such that $f(x_0) = 1$.

6. Consider a function f, defined on $[0, +\infty[$ by the expression $f(x) = \ln(3x + e^x)$.

(a) Compute $\lim\limits_{x \to +\infty} f(x)$.

(b) Compute $\lim\limits_{x \to +\infty} f'(x)$.

7. Consider an isosceles trapezium $[ABCD]$ with $\bar{BC} = \bar{AD} = 7$cm, $\bar{AB} < \bar{CD}$ and $\bar{AB} = 5$cm. Let $\alpha \in]0, \frac{\pi}{2}[$ be the amplitude, measured in radians, of the angle $D\hat{C}B$.



(a) Show that the area of $[ABCD]$ is given, in cm^2, by $A(\alpha) = 49 \sin\alpha \cos\alpha + 35 \sin\alpha$.

(b) Obtain an expression for $A'(\alpha)$.

(c) Knowing that $\tan\alpha = 5$, compute the exact value of $A(\alpha)$.

Scores

Part I

Question	1	2	3	4	5	6	7
Points	1	1	1	1	1	1	1

Part 2

Question	1	2	3a	3b	4a	4b	4c	4d	5a	5b	6a	6b	7a	7b	7c
Points	1	1	1	1	1	0.5	0.5	0.5	1	1	0.5	1	1	1	1